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Non-Discrimination and the Pareto Principle*

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1. Introduction

The principle of impartiality, or the notion of non-discrimination, is deeply rooted in many regulations, tax codes, and civil and criminal laws. To put the principle in a simple way, it typically requires that "equals should be treated equally". For the purpose of illustration, let us consider the following examples.

Example 1. A property zoning regulation has just been adopted in a community where person A and person B live. The regulation states that any resident in that community can now put fences (with specific restrictions concerning heights) around his/her yard and on his/her property. Person A is now considering whether to set up a fence around his yard (and on his property) ; so is person B. A and B are similar in so far as their incomes, sizes of their houses and yards, their jobs, etc. are concerned. They differ only in their preferences over a wooden fence and an iron fence. Person A likes to have an iron fence if possible, while person B likes to have a wooden fence if possible (assuming that both fences are within the requirements set in the regulation). Assuming all other things constant, consider the following three possible alternatives : $x=$(A with an iron fence, B with an iron fence), $y=$(A with a wooden fence, B with an iron fence), $z=$(A with an iron fence, B with a wooden fence). Notice that the alternatives x and y differ only in respect to person A's fences : an iron one or a wooden one, while the alternatives z and x differ only in respect to person B's fences : a wooden one or an iron one. In other words, person A and person B are symmetric over the pairs $\{x, y\}$ and $\{z, x\}$. If person A prefers x to y and person B prefers z to x, the property zoning regulation should treat A and B symmetrically over the pairs $\{x, y\}$ and $\{z, x\}$ respectively : if A is granted his wishes, so is B ; if B is denied her wishes, so is A.

Example 2. Household C and household D are identical as far as their taxable incomes, tax status, etc. are concerned. In filing their tax returns, they pay exactly the same amount of taxes and their tax dues will be exactly the same whether they take the standard deduction or the itemized deduction. They differ only in respect to their preferences as to which deduction to take. Household C would like to take the standard deduction, while Household D favors the itemized deduction. Assuming other things constant, consider the following three alternatives : $u=$(C taking the standard deduction, D taking the standard deduction), $v=$(C taking the itemized deduction, D taking the standard deduction), $w=$(C taking the standard deduction, D taking the itemized deduction). Note that the alternatives u and v differ only in respect to household C's taking which deduction, while w and u differ only in respect to household D's taking which deduction. In a sense, household C and household D are symmetric over the pairs $\{u, v\}$ and $\{w, u\}$. If household C prefers u to v and household D prefers w to u, the tax codes should treat them symmetrically over the pairs $\{u, v\}$ and $\{w, u\}$: if u is favored socially over v, so is w over u, and if u is not favored socially over v, so is not w over u.

Example 3. Individuals I and J are alike in many aspects, like age, marriage status, education, job, income, etc. Each unfortunately has committed a minor offense and their offenses are exactly the same in nature. For this offense, the law requires each to have 100 hours' community work or each to pay a specific amount of fine. The law permits that the offender can choose which option he/she would like to undertake. Here comes the difference between the two individuals. I would like to have 100 hours' community work, while J would like to pay the fine. Consider the following three alternatives : $a=(I$ works 100 hours for the community, J works 100 hours for the community), $b=(I$ pays the fine, J works 100 hours for the community), $c=(I$ works 100 hours for the community, J pays the fine). Note that the alternatives a and b differ only in respect to I's choices: working 100 hours for the community or paying the fine, while c and a differ only in respect to J' choices : paying the fine or working 100 hours for the community. In other words, I and J are symmetric over the pairs $\{a, b\}$ and $\{c, a\}$. If I chooses a over b and J chooses c over a, the law should treat them over their choices symmetrically : if I's choice is granted, so is J's, and if J's choice is denied, so is I's.

These examples, in part, suggest that, in many situations concerning regulations, tax laws, civil and perhaps criminal laws, there is an implicit principle requiring that there should be no discrimination against alike individuals/cases whenever individuals/cases are alike ; in other words, whenever individuals/cases are regarded as equals or similar, they should be treated equally or similarly. This principle of impartiality, or the notion of equal treatment for equals, is not an isolated idea. Indeed, it has figured prominently in the debate of a just tax system and has a long history in the utilitarian tradition. For example, Mill (1848, p. 804) defines equal treatment as follows :

> "For what reason ought equality to be the rule in matters of taxation? For the reason that it ought to be so in all affairs of government. As a government ought to make no distinction of persons or classes in the strength of their claim on it, whatever sacrifices or claims it requires from them should be made to bear as nearly as possible with the same pressure upon all, which it must be observed, is the mode by which least sacrifice is occasioned on the whole ... means equality of sacrifices."

Many years later, Sidgwick (1883, p. 562) goes further to argue that when benefit charges are not possible, "the obviously equitable principle——assuming that the existing distribution of wealth is accepted as just or not unjust ——is that equal sacrifice should be imposed on all". It is clear that the principle of equal treatment for equals is a simple consequence of the arguments put forward by both Mill and Sidgwick.

It is thus clear that the principle of impartiality or the notion of non-discrimination discussed above is so deeply rooted and fundamental in many important contexts that we do not have to give a justification here. Instead, in this paper, we try to formalize this notion of non-discrimination among alike individuals in similar situations figured in the above examples in an Arrow-Sen social choice framework, and examine its consequences in such a framework. The remainder of the paper is organized as follows. In Section 2, we introduce some basic notation and definitions as well as formalize the non-discrimination concept. Section 3 presents our result. Finally, some further discussions and a few concluding remarks are offered in Section 4.

2. The Framework

Let $X (2<|X|<\infty)$ be the set of all mutually exclusive social states. The elements of X will be denoted by x, y, z, etc. The society consists of $n (\infty>n\geq 2)$ individuals: $N=\{1, \cdots, n\}$. Each individual $i \in N$ is assumed to have a reflexive, complete and transitive binary relation (called preference ordering) R_i over X. The asymmetric and symmetric parts of R_i will be denoted by P_i and I_i respectively. Let $\wp$ be the set of all preference orderings over X and $\mathcal{D}$ be a subset of $\wp$. Let $\mathcal{D}^n$ and $\wp^n$ be the n-fold Cartesian products of $\mathcal{D}$ and $\wp$ respectively. A profile of individual preference orderings, which is an element of $\mathcal{D}^n$, is denoted by $\{R_i\}$. A social welfare function (SWF) f maps each profile $\{R_i\}$ in $\mathcal{D}^n$ to a preference ordering R (called a social preference ordering) in $\wp$. That is, $f: \mathcal{D}^n \longrightarrow \wp$. To simplify our notation, we will use R, R', etc. (with their symmetric parts I, I', etc., and asymmetric parts P, P', etc., respectively) to denote social preference orderings corresponding to individual profiles $\{R_i\}$, $\{R'_i\}$, etc.

The first condition we impose on an SWF is the familiar Unrestricted Domain condition (see Arrow (1963)).

Unrestricted Domain (UD) : The domain of an SWF, $\mathcal{D}^n$, is $\wp^n$.

The second condition imposed on an SWF is yet another familiar property known as the weak Pareto principle.

Weak Pareto Principle (WP) : For all $x, y \in X$ and all $\{R_i\} \in \mathcal{D}^n$: xPy whenever $[xP_iy$ for all $i \in N]$, where $R=f(\{R_i\})$.

Our third condition attempts to capture the idea of non-discrimination against similar individuals in similar situations in a very weak sense. It requires that there are two individuals and two distinct pairs, one for each of the two individuals, such that the society should treat the two individuals symmetrically over their respective pairs for all the individual profiles as long as the two individuals have symmetric preference orderings over the respective two pairs. Formally, our condition may be stated as follows:

Minimal Non-Discrimination (MND) : There exist two individuals $i, j \in N$ and two distinct pairs $\{x, y\}$, $\{z, w\} \subset X$ such that one and only one of the following three:
(i) $[xPy$ and $zPw]$; (ii) $[xIy$ and $zIw]$; (iii) $[yPx$ and $wPz]$, is true to hold for all $\{R_k\} \in D^n$ whenever $[xP_iy$ and $zP_jw]$, where $R=f(\{R_k\})$.

Two remarks are in order for our MND condition. First, to be concordant with the intuition provided in our three examples and the subsequent discussions in the Introduction, the pair of social states that are specified in our MND condition should be such that they differ only in the mentioned individual's personal feature (although they do not have to be confined to the individual's purely private matters). Secondly, note that in our formulation of MND, a kind of Arrow type independence condition is embedded implicitly. Explicitly, our MND condition requires that, given that $\{x, y\}$ is individual i's pair and $\{z, w\}$ is individual j's pair, if, for two preference profiles, $\{R_k\}$ and $\{R_k'\} \in \mathcal{D}^n$, xP_iy and zP_jw, and $xP_i'y$ and $zP_j'w$, then xRy iff $xR'y$, and zRw iff $zR'w$, where $R=f(\{R_k\})$ and $R'=f(\{R'_k\})$. This independence condition is by no means unique to MND. Indeed, a similar type of independence condition is implicitly embedded in the weak Pareto principle as well.

3. The Result

In this section, we show that the three conditions proposed in the last section are incompatible for an SWF.

Theorem. There exists no SWF satisfying UD, WP and MND.

Proof. Suppose that there are two individuals, i and j, and two distinct pairs, $\{x, y\}$ and $\{z, w\}$, such that xP_iy and zP_jw. Suppose to the contrary that there exists an SWF satisfying UD, WP and MND. We will derive certain contradictions to prove our result. For x, y, z, w, there are two cases: (i) $\{x, y\}$ and $\{z, w\}$ have one element in common; (ii) x, y, z, w are distinct.

(i) Without loss of generality, in this case, we take $w=x$. Thus, we have xP_iy and zP_jx. Consider the social preference ordering over x and y. One and only one of the following three possibilities can happen: (i.1) xPy; (i.2) yPx; and (i.3) xIy. In case (i.1), by UD, consider yP_kz for all $k \in N$. By MND, we have zPx. By WP, we obtain yPz. Thus, we have xPy, yPz and zPx, a cycle which contradicts the transitivity of R. In case (i.2), by UD, consider zP_ky for all $k \in N$. By MND, xPz; by WP, zPy. Thus, we obtain xPz, zPy and yPx; a contradiction. Finally, in case (i.3), by UD, consider yP_kz for all $k \in N$. By MND, zIx; by WP, yPz. Thus, we have zIx, yPz and xIy; a contradiction.

Therefore, in this case, case (i), there is no SWF satisfying UD, WP and MND.

(ii) Consider the social preference ordering over x and y. Again, we can have one and only one of the following three possibilities: (ii.1) xPy; (ii.2) yPx; (ii.3) xIy. In case (ii.1), by UD, consider yP_kz and wP_kx for all $k \in N$. Then, by MND, we have zPw; by WP, we have yPz and wPx. Thus, we obtain xPy, yPz, zPw and wPx; a contradiction. In case (ii.2), by UD, consider zP_ky and xP_kw for all $k \in N$. Then, by MND, wPz; by WP, zPy and xPw. Thus, we obtain xPw, wPz, zPy and yPx; a contradiction. Finally, in case (ii.3), by UD, consider yP_kz and wP_kx for all $k \in N$. By MND, wIz; by WP, yPz and wPx. Thus, we obtain yPz, zIw, wPx and xIy; a contradiction.

Therefore, in case (ii), there is no SWF satisfying UD, WP and MND. Combining (i) and (ii), the theorem is proved. ■

4. Discussion and Conclusion

As our theorem suggests, there is a fundamental conflict between the (weak) Pareto principle and the notion of non-discrimination. To see how the conflict may arise in our examples of Section 1, let us consider Example 1. In the example, we have two persons A and B and three alternatives $x=$(A with an iron fence, B with an iron fence), $y=$(A with a wooden fence, B with an iron fence), $z=$(A with an iron fence, B with a wooden fence). As we have demonstrated in Section 1, person A and person B are symmetric over the pairs $\{x, y\}$ and $\{z, x\}$. Note that person A prefers x to y, while person B prefers z to x. How should the society (regulation) treat person A and person B with respect to their corresponding symmetric pairs? Well, there are exactly three mutually exclusive possibilities according to MND: (i) preserving each person's preferences over their respective symmetric pairs, that is, x is socially preferred to y and z is socially preferred to x; (ii) declaring social indifference for their respective symmetric pairs, i.e., x is socially indifferent to y and z is socially indifferent to x; and (iii) reversing each person's preferences over their respective symmetric pairs, that is, y is socially preferred to x and x is socially preferred to w. Now, let us analyse each case separately.

(i) In this case, xPy and zPx. Consider the following preference orderings of persons A and B. For A, the thinking is this: "The best scenario for me is to see both of us have iron fences; but if that is not possible, I'd like to see that B gets an iron fence more than I get an iron fence." As a consequence, A's ranking of x, y, z is: x best, y second and z last. For B, she has the following reasoning: "I hate to see that A has an iron fence; however, other things the same, I'd like my fence to be wooden." Given this reasoning, B ranks x, y, z as follows: y best, z second and x last. Note that A ranks x above y and B ranks z above x. Note also that both A and B rank y above z, by the weak Pareto principle (assuming that all other individuals concerned also rank y above z), y is socially preferred to z. Thus, we end up with xPy, zPx and yPz, a cycle.

(ii) In this case, xIy and zIx. We can tell exactly the same story as in (i) and as a consequence, A has the following ranking: x first, y second, and z last, and B ranks y first, z second and x last. Again, by WP, yPz. Thus, we obtain xIy, zIx and yPz, an intransitivity of social preference ordering.

(iii) In this case, yPx and xPz. Now, consider the following preference orderings for persons A and B. For A, the situation is: "I really do not like that my fence is wooden while B's fence is iron!". Thus, A ranks both x and z above y. For B, the situation is this: "Oh, I love to see that A has an iron fence and I have a wooden one!" As a consequence, B places z above both x and y. Notice first that A prefers x to y, while B prefers z to x. Secondly, notice that both A and B place z above y. Hence, by WP (assuming all other individuals concerned prefer z to y), z is socially preferred to y. Therefore, we have the following social preference ordering: yPx, xPz and zPy, a cycle.

Thus, in each case, there is a profile of individual preference orderings such that the weak Pareto principle is in conflict with the minimal non-discrimination principle if all logically possible individual preference profiles are allowed.

Our theorem does not require Arrow's much more demanding independence of irrelevant alternatives condition (see Arrow (1963)). The rationality condition on the social preference ordering is the same as Arrow's requirement, viz., a transitive social preference ordering. The weak Pareto principle is in its weakest form. The non-discrimination condition, we believe, is in a weak form as well: the social welfare function should treat two similar individuals similarly over their symmetric pairs of social states if the two individuals have symmetric preference orderings over their respective pairs[1]. It is thus a very minimal demand for a non-discriminatory and impartial social welfare function. Yet, combined with the Pareto principle, we face a dilemma.

The dilemma seems to suggest that, in designing regulations, laws, and public policies[2], we may encounter the occasional incompatibility of the two seemingly appealing principles : the Pareto principle and the principle of non-discrimination. This dilemma is comparable to Sen's Paretian libertarian paradox (Sen (1970)) where he shows that there is a fundamental conflict between the Pareto principle and individual rights. Note that our MND condition is both formally and conceptually weaker than Sen's minimal libertarian (ML) condition which requires that there are two individuals i and j, and two distinct pairs $\{x, y\}$ and $\{a, b\}$, $\{x, y\}$ for i and $\{a, b\}$ for j, such that each is decisive over his/her respective pair : xPy whenever xP_iy and aPb whenever aP_jb[3]. Note also that Sen's condition ML implicitly assumes that in certain circumstances concerning private matters of the concerned individuals, information about preferences of other individuals over $\{x, y\}$ and $\{a, b\}$ is not relevant in judging whether i's circumstances in the context of $\{x, y\}$ is similar to j's circumstances in the context of $\{a, b\}$. Our condition MND, on the other hand, does not require such a demanding assumption. Indeed, in our framework, information about preferences of other individuals over the respective pairs of $\{x, y\}$ and $\{z, w\}$ figured into the condition MND may be relevant in deciding social preferences over $\{x, y\}$ and $\{z, w\}$: the only constraint here is that the social preferences over $\{x, y\}$ and $\{z, w\}$ are linked in a particular way as specified in the condition. As our theorem suggests, there is even a conflict between the Pareto principle and the value of justice reflected in the minimal non-discrimination condition.

It should be noted that if each pair of social states that figure in our MND condition is confined to be concerned with the respective individual's purely private matters, then the condition MND is a weakening of Sen's minimal libertarian condition. Viewed in this fashion, our result can be regarded as an extension of Sen's liberal paradox[4].

Finally, as we remarked in Section 2, our condition MND implicitly assumes a kind of Arrow type independence condition and this type of independence condition is also implicitly embedded in the Pareto principle. The nature of this type of independence condition has powerful implications as noted by Sen (1976) that the Pareto principle has some "epidemic" properties for a social welfare function with unrestricted domain. If an individual is decisive over a pair, this individual will have a weak form of decisiveness over every ordered pair. The Paretian epidemic thus leaves no room for the social preference ordering over the respective pairs of the concerned individuals figured in MND condition to be strict. It is exactly this nature of the Pareto principle that leads to the impossibility result of the present paper.

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Notes

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1) The principle of non-discrimination can be regarded as a notion of justice. There are two distinct approaches to the issue of justice. The first and more traditional one focuses on the *end-states* or the *outcomes*. Justice is viewed as a reflection of the nature of outcomes as well as individual preferences over the outcomes. The second approach, which is due to Nozick (1974), emphasizes on the *process* of arriving at the outcomes. It is then argued that justice is captured by the nature of the process and has little to do with the outcomes and with individual's preferences over the outcomes. Confronted with these two distinct views of justice, in this paper, I have adopted the outcome-based approach. However, if one favors the process-based approach, he/she may formulate the principle of non-discrimination in such a framework. It should be noted that the problem of different approaches to the same issue is not unique to our case. Indeed, in the discussion of modelling individual rights which are closely related to our notion of non-discrimination, there are two distinct approaches that are similar to the approaches to the issue of justice we have just outlined. See, for example, Gaertner, Pattanaik and Suzumura (1992) and Sen (1992) for discussions of relative advantages and disadvantages of each approach.

2) In the public finance literature, there is a tradition of using horizontal equity and vertical equity to design tax policies (see Musgrave (1959)). Horizontal equity often states that equals should be treated equally. Thus, our notion of non-discrimination corresponds to horizontal equity.

3) Since our MND condition is both formally and conceptually weaker than Sen's ML condition, anyone who objects to ML need not necessarily object to our MND condition. In this respect, our impossibility result suggests a more fundamental and deeper conflict between the Pareto Principle and the notion of non-discrimination.

4) Note, however, the rationality condition required in Sen's Paretian libetarian paradox is an acyclic social preference relation. The acyclicity of the social preference relation is a weaker property than the transitivity of the social preference relation. It is worth noting that it is possible to construct a social decision function, which maps each profile $\{R_i\}$ in $\mathcal{D}^n$ to an acyclic binary relation over X, that satisfies the weak Pareto principle and the minimal non-discrimination condition.

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