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特集　計量経済学の方法と応用

A System Method of Prediction in Simultaneous Equations

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Abstract

This paper extends the single equation method of prediction in a simultaneous equations system by Takeuchi (1972) to a system method of prediction associated with the full information maximum likelihood (FIML) estimator. We derive the asymptotic expansion of the bias and the mean square error of the proposed predictor to terms of order T^{-1} (T is the sample size).

1. Introduction

Less attention has been paid to the problem of predicting the endogenous variables in a simultaneous equations system comapared with the problem of estimating the structural parameters. Klein (1960) argued that the derived estimates of the reduced form coefficients will be more efficient than those obtained with no restriction procedure, consequently the prediction based on the former will be more efficient than that based on the latter. Goldberger, Nagar and Odeh (1961) gave the asymptotic covariance matrix of the two stage least squares (TSLS) induced restricted reduced form estimator. Nagar and Sahay (1978) derived the exact and large noncentrality parameter asymptotic moments of the partially restricted reduced form predictor. This method was originally proposed by Amemiya (1966). Takeuchi (1972) proposed a predictor which is obtained by the use of the maximum likelihood estimate of the reduced form coefficents taking account of the overidentifying restrictions imposed on a relevant structural equation. This method is called the single equation method of prediction (SEMP) in analogy with the single equation method of estimation for the structural parameters.

Tsukuda (1989) derived the asymptotic expansions of the bias and the mean square error (m. s. e.) of the SEMP associated with the limited information maximum likelihood (LIML) and the TSLS estimators when the sample size increases. Both the LIML and the TSLS predictors asymptotically dominate the partially restricted reduced form predictor as well as the unrestricted reduced form predictor for any values of parameters.

The purpose of this paper is to extend the results of Tsukuda (1989) to the system method of prediction associated with the FIML estimator. We will derive the asymptotic bias and the m.s.e. of the predictor to terms of order T^{-1} (T is the sample size). Morimune (1988) proposed a modified three stage least squares (M3SLS) estimator which is third order efficient, and shares the same asymptotic expansion of the distribution as the FIML estimator to terms of order T^{-1}. Hence, the predictor associated with the M3SLS estimator has the same asymptotic bias and the m.s.e. as those of the FIML predictor.

2. Predictors and Results

Let a system of structural equations be

(2.1)　$YB+Z\Gamma=U,$

and the reduced form equations be

(2. 2) $Y=Z\Pi+V$

where Y and Z are $T\times G$ and $T\times K$ matrices of observations on the endogenous and exogenous variables, respectively, B and Γ are $G\times G$ and $K\times G$ matrices of unknown coefficients, some of whose elements are known to be zeros. V and U are $T\times G$ matrices of disturbances and

(2. 3) $V=UB^{-1}$,

each row of which is independently distributed as $N(0, \Omega)$. The structural form error covarince matrix is denoted as Σ. The reduced form coefficient are

(2. 4) $\Pi=-\Gamma B^{-1}$.

Each equation in the system (2. 1) may be written as

(2. 5) $y_i=Y_i\beta_i+Z_i\gamma_i+u_i,\ i=1, \cdots, G.$

Here coefficients β_i and $\gamma_i (i=1, \cdots, G)$ have opposite sign from B and Γ. A vector of all coefficients is defined to be $\theta'=(\beta'_1, \gamma'_1, \cdots, \beta'_{G'}\gamma'_G)'$ that is $S\times 1$ and S is equal to the total number of coefficients $(\Sigma_{i=1,\ G}(G_i+K_i))$. It is assumed that all of structural equations in the system are identified. The reduced form coefficients are a function of θ, and are explicitly written as $\Pi=\Pi(\theta)$.

For a given set of K-exogenous variables z_0, let

(2. 6) $y_0=z'_0\Pi+v_{0'}$

where a $1\times G$ disturbance vector v_0 is distributed as $N(0, \Omega)$ and is assumed to be independent of V. The problem of this paper is to predict the expected values of y_0, namely to predict

(2. 7) $E\{y_0\}=z'_0\Pi.$

A system method of prediction is defined by

(2. 8) $\hat{y}=z'_0\Pi(\hat{\theta}),$

where $\hat{\theta}$ is either the FIML or the M3SLS estimator.

The FIML estmator of B and Γ is derived by maximizing $-(T/2)\log|(1/T)(Y+Z\Gamma B^{-1})'(Y+Z\Gamma B^{-1})|$ with respect to B and Γ. The M3SLS estimator proposed by Morimune (1988) is defined as follows. The 3SLS estimator is well known and symbolically written as

(2. 9) $$\hat{\theta}_{3SLS}=\left[\hat{\sigma}^{ij}\begin{pmatrix}\hat{\Pi}'_iZ'Z\hat{\Pi}_j & \hat{\Pi}'_iZ'_j\\ Z'_iZ\hat{\Pi}_j & Z'_iZ_j\end{pmatrix}\right]^{-1}\Sigma_{k=1,\ G}\left[\hat{\sigma}^{ik}\begin{pmatrix}\hat{\Pi}'_jZ'y_k\\ Z'_jy_k\end{pmatrix}\right].$$

The error covariance matrix Σ is estimated by the 2SLS method, and $\hat{\sigma}^{ij}(i, j=1, \cdots, G)$ is the ith and jth element in $\hat{\Sigma}^{-1}$, $\hat{\Pi}_i$ is the proper subcolumns of the ordinary least squares estimator of the reduced form coefficients

(2. 10) $\hat{\Pi}=(Z'Z)^{-1}Z'Y.$

The M3SLS is defined similarly as the 3 SLS, and is characterized as a four stage least squares: first the usual 3 SLS estimator is calculated then Σ is reestimated by using the 3SLS estimator, and Π is reestimated by the 3SLS estimator of B and Γ, i.e., $\tilde{\Pi}=-\hat{\Gamma}_{3SLS}\hat{B}^{-1}{}_{3SLS}$. The M3SLS is defined to be

(2. 11) $$\hat{\theta}_{M3SLS}=\left[\hat{\sigma}^{ij}{}_{3SL}\begin{pmatrix}\tilde{\Pi}'_iZ'Z\hat{\Pi}_j\\ Z'_iZ\,\hat{\Pi}_j\end{pmatrix}\right.$$

$$\left.\begin{matrix}\tilde{\Pi}_i Z' Z_j \\ Z'_i Z_j\end{matrix}\right)\Bigg]^{-1} \Sigma_{k=1,\,G}\left[\hat{\sigma}^{ik}{}_{3SL}\begin{pmatrix}\tilde{\Pi}'_j Z' y_k \\ Z'_j y_k\end{pmatrix}\right],$$

where $\tilde{\Pi}_i$ consists of proper subcolumns in $\tilde{\Pi}$ corresponding to the reduced form coefficients for Y_i.

Before stating the results, we introduce some notations for convenience of expositions:

(2.12) $M=\frac{1}{T}Z'Z,$

(2.13) $F_{\alpha\beta}=tr\left\{\Omega^{-1}\frac{\partial\Pi'}{\partial\theta_\alpha}M\frac{\partial\Pi}{\partial\theta_\beta}\right\},$

(2.14) $J_{\alpha\beta.\gamma}=tr\left\{\Omega^{-1}\frac{\partial^2\Pi'}{\partial\theta_\alpha\partial\theta_\beta}M\frac{\partial\Pi}{\partial\theta_\gamma}\right\},$

(2.15) $L_{\alpha\beta\gamma.\delta}=tr\left\{\Omega^{-1}\frac{\partial^3\Pi'}{\partial\theta_\alpha\partial\theta_\beta\partial\theta_\gamma}M\frac{\partial\Pi}{\partial\theta_\delta}\right\},$

(2.16) $M_{\alpha\beta.\gamma\delta}=tr\left\{\Omega^{-1}\frac{\partial^2\Pi'}{\partial\theta_\alpha\partial\theta_\beta}M\frac{\partial^2\Pi}{\partial\theta_\gamma\partial\theta_\delta}\right\},$

(2.17) $W_{\alpha\beta\gamma\delta}=tr\left\{\Omega^{-1}\frac{\partial\Pi'}{\partial\theta_\alpha}M\frac{\partial\Pi}{\partial\theta_\beta}\Omega^{-1}\frac{\partial\Pi'}{\partial\theta_\gamma}M\frac{\partial\Pi}{\partial\theta_\delta}\right\}.$

The $s\times s$ matrix F, whose α-βth element is $F_{\alpha\beta}$, is the asymptotic Fisher information matrix of θ. The $\alpha-\beta$th element of F^{-1} is denoted as $F^{\alpha\beta}$.

Let us define the normalized error of the predictor as

(2.18) $\hat{e}_y=\sqrt{T}(\hat{y}-z'_0\Pi)$
$=\sqrt{T}z'_0(\Pi(\hat{\theta})-\Pi(\theta)).$

Next theorems state the asymptotic bias and the m.s.e of the system method of prediction associated with the FIML and the M3SLS estimators, to terms of order T^{-1}.

Theorem 1: As $T\to\infty$, the asymptotic bias of the system method of prediction is given, to terms of order T^{-1}, by

(2.19) $AM_T\{\hat{e}_y\}$

$$=\frac{1}{\sqrt{T}}z'_0\left\{\Sigma_{\alpha,\beta,\gamma,\delta=1,s}\frac{\partial\Pi}{\partial\theta_\delta}F^{\alpha\beta}F^{\gamma\delta}tr\left[\frac{\partial(B'\Gamma')'}{\partial\theta_\alpha}B^{-1}\frac{\partial B}{\partial\theta}\Sigma^{-1}\frac{\partial(B'\Gamma')}{\partial\theta_\gamma}(\Pi\ I)'M(\Pi\ I)\right]+\frac{1}{2}\Sigma_{\alpha,\beta=1,s}\frac{\partial^2\Pi}{\partial\theta_\alpha\partial\theta_\beta}F^{\alpha\beta}\right\}.$$

Theorem 2: As $T\to\infty$, the asymptotic m.s.e. of the system method of predition is given, to terms of order T^{-1}, by

(2.20) $AM_T\{\hat{e}_y\hat{e}'_y\}$

$$=\Sigma_{i,j=1,s}z'_0\frac{\partial\Pi}{\partial\theta_i}\frac{\Pi'}{\partial\theta_j}z_0F^{ij}+\frac{1}{T}\left\{\Sigma_{i,j,k,l=1,s}z'_0\left(\frac{1}{4}\frac{\partial^2\Pi}{\partial\theta_i\partial\theta_j}\frac{\partial^2\Pi'}{\partial\theta_k\partial\theta_l}+\frac{1}{3}\frac{\partial^3\Pi}{\partial\theta_i\partial\theta_j\partial\theta_k}\cdot\frac{\partial\Pi'}{\partial\theta_l}\right)z_0H_{ijkl}\right.$$
$$-\frac{1}{2}\Sigma_{i,j,k=1,s}z'_0\left(\frac{\partial\Pi}{\partial\theta_i}\frac{\partial^2\Pi'}{\partial\theta_j\partial\theta_k}+2\frac{\partial^2\Pi}{\partial\theta_i\partial\theta_j}\frac{\partial\Pi'}{\partial\theta_k}\right)z_0\cdot\Sigma_{\alpha,\beta,\gamma,=1,s}F^{i\alpha}J_{\beta\gamma.\alpha}H_{\beta\gamma jk}+\Sigma_{i,j=1,s}z'_0\frac{\partial\Pi}{\partial\theta_i}\frac{\Pi'}{\partial\theta_j}z_0\left\{(G+1)F^{ij}\right.$$
$$-\frac{1}{3}\Sigma_{\alpha,\beta,\gamma,\delta=1,s}F^{i\alpha}[L_{\beta\gamma\delta.\alpha}+3M_{\alpha\beta.\gamma\delta}-3\Sigma_{o,p=1,s}(J_{\alpha\beta.o}+J_{\beta o.\alpha})J_{\gamma\delta.o}F^{op}]H_{\beta\gamma\delta j}$$
$$\left.\left.-\Sigma_{\alpha,\beta=1,s}F^{i\alpha}\left[\Sigma_{\gamma,\delta=1,s}(2W_{\alpha\delta\beta\gamma}-3M_{\alpha\delta.\beta\gamma})F^{\delta\gamma}+\Sigma_{\gamma,\delta,o,p=1,s}\left(3J_{\alpha\beta.\gamma}J_{\delta p.o}F^{\beta p}F^{\gamma o}-\frac{1}{4}J_{\beta\gamma.\alpha}J_{op.\delta}\cdot H_{\beta\gamma op}\right)\right]F^{\beta j}\right\}\right\},$$

where

$$H_{ijkl}=F^{ij}F^{kl}+F^{ik}F^{jl}+F^{il}F^{jk}.$$

When the structural coefficients are

(2.21) $(B'\Gamma')=\begin{pmatrix}1 & -\beta'_1 & -\gamma'_1 & 0\\ 0 & I & & \Pi'_1\end{pmatrix},$

the system method of prediction reduces to the single equation method of prediction associated with the LIML estimator. Theorems 1 and 2 for this case reduce to the theorems 2 and 3 in Tsukada (1989).

3. Proofs of Theorems

Since the derivations of Theorems 1 and 2 heavily depend upon Morimune (1988), we use the same symbols and notations as those of him. The equations (3.19), (3.20) and (3.21) in Morimune (1988) give each term of the stochastic expansion of $\hat{e}_{\mathrm{FIML}}=\sqrt{T}(\hat{\theta}_{\mathrm{FIML}}-\theta)$, to terms of order T^{-1}, as

$$(3.1)\quad \hat{e}_{\mathrm{FIML}}=e^{(0)}+\frac{1}{\sqrt{T}}e^{(1)}+\frac{1}{T}e^{(2)}+O_p(T^{-3/2}).$$

Morimune (1988, equations (4.1), (4.3), (4.4), (4.5) and (4.18)) also state the conditional expectations of $e^{(1)}$, $e^{(2)}$ and $e^{(1)}e^{(1)\prime}$, given $e^{(0)}$ fixed, as

$$(3.2)\quad E(e^{(1)}|e^{(0)})=-\frac{1}{2}\sum_{\alpha,\beta,\gamma=1,s}F^{-1}l_{\alpha}.J_{\beta\gamma.\alpha}(e_{\beta}^{(0)}e_{\gamma}^{(0)}),$$

$$(3.3)\quad E(e^{(2)}|e^{(0)})=-\sum_{\alpha=1,s}F^{-1}l_{\alpha}\Big\{\sum_{\beta,\gamma=1,s}(J_{\alpha\beta.\gamma}+J_{\beta\gamma.\alpha})e_{\beta}^{(0)}E(e_{\gamma}^{(1)}|e^{(0)})+\frac{1}{6}\sum_{\beta,\gamma,\delta=1,s}(L_{\beta\gamma\delta.\alpha}+3M_{\alpha\beta.\gamma\delta})(e_{\beta}^{(0)}e_{\gamma}^{(0)}e_{\delta}^{(0)})+\sum_{\beta,\gamma,\delta,o,p=1,s}e_{\gamma}^{(0)}J_{\gamma\beta.\delta}F^{\delta o}J_{\alpha p.o}F^{p\beta}-\sum_{\beta,\gamma,\delta=1,s}M_{\alpha\delta.\gamma\beta}e_{\gamma}^{(0)}F^{\delta\beta}\Big\}.$$

$$(3.4)\quad E(e^{(1)}e^{(1)\prime}|e^{(0)})=(G+1)F^{-1}-\sum_{\alpha,\beta,\gamma,\delta=1,s}F^{-1}l_{\alpha}(W_{\alpha\beta\gamma\delta}+W_{\alpha\beta\delta\gamma})F^{\alpha\gamma}l'_{\delta}F^{-1}+\sum_{\alpha,\beta,\gamma,\delta=1,s}F^{-1}l_{\alpha}.M_{\alpha\beta\gamma\delta}.l'_{\delta}F^{-1}(e_{\beta}^{(0)}e_{\gamma}^{(0)})-\sum_{\alpha,\beta,\delta,o,p=1,s}F^{-1}l_{\alpha}.J_{\alpha\beta.\gamma}F^{\gamma\delta}J_{op.\delta}.l'_{0}F^{-1}(e_{\beta}^{(0)}e_{p}^{(0)})+\frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta,o,p=1,s}F^{-1}l_{\alpha}J_{\gamma\beta.\alpha}J_{op.\delta}.l'_{o}F^{-1}(e_{\beta}^{(0)}e_{\gamma}^{(0)}e_{o}^{(0)}e_{p}^{(0)}).$$

The stochastic expansion of the reduced form FIML estimator $\hat{E}=\sqrt{T}(\Pi(\hat{\theta}_{\mathrm{FILM}})-\Pi(\theta))$, to terms of order T^{-1}, is given by

$$(3.5)\quad \hat{E}=E^{(0)}+\frac{1}{\sqrt{T}}E^{(1)}+\frac{1}{T}E^{(2)}+O_p(T^{-3/2}),$$

where

$$(3.6)\quad E^{(0)}=\sum_{\alpha=1,s}\frac{\partial\Pi}{\partial\theta_{\alpha}}e_{\alpha}^{(0)},$$

$$(3.7)\quad E^{(1)}=\sum_{\alpha=1,s}\frac{\partial\Pi}{\partial\theta_{\alpha}}e_{\alpha}^{(1)}+\frac{1}{2}\sum_{\alpha,\beta=1,s}\frac{\partial^2\Pi}{\partial\theta_{\alpha}\partial\theta_{\beta}}(e_{\alpha}^{(0)}e_{\beta}^{(0)}),$$

$$(3.8)\quad E^{(2)}=\sum_{\alpha=1,s}\frac{\partial\Pi}{\partial\theta_{\alpha}}e_{\alpha}^{(2)}+\frac{1}{2}\sum_{\alpha,\beta=1,s}\frac{\partial^2\Pi}{\partial\theta_{\alpha}\partial\theta_{\beta}}\{e_{\alpha}^{(1)}e_{\beta}^{(0)}+e_{\alpha}^{(0)}e_{\beta}^{(1)}\}+\frac{1}{6}\sum_{\alpha,\beta,\gamma=1,s}\frac{\partial^3\Pi}{\partial\theta_{\alpha}\partial\theta_{\beta}\partial\theta_{\gamma}}(e_{\alpha}^{(0)}e_{\beta}^{(0)}e_{\gamma}^{(0)}).$$

The stochastic expansion of $\hat{e}_y$ is given by

$$(3.9)\quad \hat{e}_y=e_y^{(0)}+\frac{1}{\sqrt{T}}e_y^{(1)}+\frac{1}{T}e_y^{(2)}+O_p(T^{-3/2}),$$

where $e_y^{(i)}=z'_0E^{(i)}(i=0,1,2)$. Using the formulas of (3.2), (3.3) and (3.4), we can evaluate the asymptotic moments of $\hat{e}_y$. The asymptotic bias is given by

$$(3.10)\quad \mathrm{AM_T}\{\hat{e}_y\}=\frac{1}{\sqrt{T}}z'_0E\{E^{(1)}\}=\frac{1}{\sqrt{T}}z'_0\Big\{\sum_{\alpha=1,s}\frac{\partial\Pi}{\partial\theta_{\alpha}}E\{e_{\alpha}^{(1)}\}+\frac{1}{2}\sum_{\alpha,\beta=1,s}\frac{\partial^2\Pi}{\partial\theta_{\alpha}\partial\theta_{\beta}}E\{e_{\alpha}^{(0)}e_{\beta}^{(0)}\}\Big\},$$

which turns out to (2.19) in Theorem 1. The asymptotic m.s.e. of $\hat{e}_y$ is given by

$$(3.11)\quad AM_T\{\hat{e}_y\hat{e}'_y\}=E\{e_y^{(0)}e_y^{(0)\prime}\}$$

$$+\frac{1}{T}E\{e_y^{(1)}e_y^{(1)\prime}+2e_y^{(2)}e_y^{(0)\prime}\}.$$

After some calculations Theorem 2 follows.

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