経済研究
Vol. 41, No. 3, Jul. 1990

特集　計量経済学の方法と応用

Asymptotic Distribution of the Least Squares Estimator of the Cointegrating Vector

Katsuto Tanaka

1. Introduction

Problems associated with cointegration have recently attracted much attention. Granger (1981) first pointed out that integrated, multiple time series may have linear combinations which are stationary. In such a case those variables are said to be cointegrated and the transformation matrix which makes the integrated process stationary is called the cointegrating matrix. Engle and Granger (1987) discuss from a practical point of view the estimation and testing procedures for cointegration. Phillips and Ouliaris (1987) develop an asymptotic theory for tests for the presence of cointegration. Engle and Yoo (1987) explore the multistep forecasting behavior of cointegrated processes.

In this paper we concentrate primarily on the regression relation among components of the integrated process and obtain the asymptotic distribution of the least squares estimator (LSE) of the regression coefficient. Especially for the two dimensional case we compute the limiting distribution of the normalized estimator. In Section 2 we describe a general model for the integrated process. In Section 3 we discuss the case where there exists no cointegration. This includes as a special case the spurious regression observed in Granger and Newbold (1974). Section 4 deals with the case of cointegration. In these two sections the limiting distribution functions are obtained together with percent points and moments. Proofs of some theorems are provided in Appendix.

2. Model and Assumptions

We assume that the q-dimensional integrated process $\{y_j\}$ is generated by

$$(1) \quad y_j = y_{j-1} + u_j, \qquad y_o = 0,$$

where $\{u_j\}$ is a stationary process defined by

$$(2) \quad u_j = \sum_{l=0}^{\infty} C_l \varepsilon_{j-l}, \qquad \sum_{l=0}^{\infty} l \| C_l \| < \infty,$$
$$A \equiv \sum_{l=0}^{\infty} C_l \neq 0.$$

Here $\|M\|$ denotes the square root of the largest eigenvalue of $M'M$ and $\{\varepsilon_j\}$ is a sequence of i. i. d. $(0, I_q)$ random variables with I_q being the $q \times q$ identity matrix. The main purpose here is to study asymptotic properties of regression relations among the components of $\{y_j\}$. For this purpose we decompose y_j, ε_j and A into

$$(3) \quad y_j = \begin{pmatrix} y_{1j} \\ y_{2j} \end{pmatrix} \begin{matrix} \updownarrow \ q_1 \\ \updownarrow \ q_2 \end{matrix}, \qquad \varepsilon_j = \begin{pmatrix} \varepsilon_{1j} \\ \varepsilon_{2j} \end{pmatrix} \begin{matrix} \updownarrow \ q_1 \\ \updownarrow \ q_2 \end{matrix},$$
$$A = \begin{pmatrix} A'_1 \\ A'_2 \end{pmatrix} \begin{matrix} \updownarrow \ q_1 \\ \updownarrow \ q_2 \end{matrix},$$

where we assume that

$$(4) \quad \mathrm{rank}(A'_1) = q_1.$$

Let us now consider the regression relations

(5) $y_{2j}=\hat{B}y_{1j}+\hat{u}_{2j}$,
(6) $y_{2j}=\tilde{\alpha}+\tilde{B}y_{1j}+\tilde{u}_{2j}$,

where

$$\hat{B}=\sum_{j=1}^{T}y_{2j}y'_{1j}\left(\sum_{j=1}^{T}y_{1j}y'_{1j}\right)^{-1},$$

$$\tilde{\alpha}=\bar{y}_2-\tilde{B}\bar{y}_1,\quad \bar{y}_1=\frac{1}{T}\sum_{j=1}^{T}y_{1j},\quad \bar{y}_2=\frac{1}{T}\sum_{j=1}^{T}y_{2j},$$

$$\tilde{B}=\sum_{j=1}^{T}(y_{2j}-\bar{y}_2)(y_{1j}-\bar{y}_1)'\left(\sum_{j=1}^{T}(y_{1j}-\bar{y}_1)(y_{1j}-\bar{y}_1)'\right)^{-1}.$$

Note that $\hat{B}$ and $\tilde{B}$ are the LSE' s without and with mean correction respectively.

In the next section we discuss asymptotic distributions of $\hat{B}$ and $\tilde{B}$ when the matrix A in (2) is nonsingular. The case when A is singular corresponds to cointegration and is discussed in Section 4.

3. Nonsingular Case-No Cointegration

In this section we assume that the matrix A in (2) is nonsingular. This is equivalent to saying that the spectral matrix $f_u(\omega)$ of $\{u_j\}$ evaluated at the origin is nonsingular since $f_u(0)=AA'/(2\pi)$. To obtain the asymptotic distributions of $\hat{B}$ and $\tilde{B}$ in (5) and (6) respectively we need the following lemma, which is proved following the same lines as in Tanaka (1988).

Lemma 1 : As $T\to\infty$, it holds that

$$L\left(\frac{1}{T^2}\sum_{j=1}^{T}y_jy'_j\right)\to L(AW_MA'),$$

$$L\left(\frac{1}{T^2}\sum_{j=1}^{T}(y_j-\bar{y})(y_j-\bar{y})'\right)\to L(AW_BA'),$$

where $L(X)$ denotes the probability law of X while

$$W_M=\int_0^1\int_0^1[1-\max(s,t)]\,dw(s)\,dw(t)',$$

$$W_B=\int_0^1\int_0^1[\min(s,t)-st]\,dw(s)\,dw(t)',$$

with $\{w(t)\}$ being q-dimensional Brownian motion with $E(w(t))=0$ and $E(w(s)w(t)')=\min(s,t)I_q$.

Phillips and Durlauf (1986) used different, but equivalent expressions for W_M and W_B, which are

$$L(W_M)=L\left(\int_0^1 w(t)w(t)'dt\right),$$

$$L(W_B)=L\left(\int_0^1 w(t)w(t)'dt-\int_0^1 w(t)dt\int_0^1 w(t)'dt\right).$$

For the derivation of distribution functions as described in Theorem 2 below the present expressions are more convenient. Using Lemma 1 we now have the following theorem concerning the weak convergence of $\hat{B}$ and $\tilde{B}$, which is proved from Lemma 1 using the continuous mapping theorem.

Theorem 1 : As $T\to\infty$, it holds that

$$L(\hat{B})\to L(A'_2W_MA_1(A'_1W_MA_1)^{-1}),$$

$$L(\tilde{B})\to L(A'_2W_BA_1(A'_1W_BA_1)^{-1}).$$

Note that $A'_1W_MA_1$ and $A'_1W_BA_1$ are nonsingular with probability 1 because of the assumption (4).

Let us restrict ourselves to the case $q_1=q_2=1$ so that $\hat{B}$ and $\tilde{B}$ are scalar and put

$$F_1(x)=\lim_{T\to\infty}P(\hat{B}<x)=P\left(xA'_1W_MA_1-\frac{1}{2}(A'_2W_MA_1+A'_1W_MA_2)>0\right),$$

$$F_2(x)=\lim_{T\to\infty}P(\tilde{B}<x)=P\left(xA'_1W_BA_1-\frac{1}{2}(A'_2W_BA_1+A'_1W_BA_2)>0\right).$$

Then F_1 and F_2 can be calculated from the following theorem.

Theorem 2 :

$$F_j(x)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{1}{\theta}\,\mathrm{Im}(\varphi_j(\theta))\,d\theta, \qquad (j=1,2),$$

where $\mathrm{Im}(Z)$ is the imaginary part of Z and

$$\varphi_1(\theta)=[\cos\sqrt{2i\theta\xi_1(x)}\,\cos\sqrt{2i\theta\xi_2(x)}]^{-\frac{1}{2}},$$

$$\varphi_2(\theta)=\left[\frac{\sin\sqrt{2i\theta\xi_1(x)}}{\sqrt{2i\theta\xi_1(x)}}\,\frac{\sin\sqrt{2i\theta\xi_2(x)}}{\sqrt{2i\theta\xi_2(x)}}\right]^{-\frac{1}{2}},$$

$$\xi_1(x),\,\xi_2(x)=\frac{1}{2}(A'_1(xA_1-A_2)\pm\sqrt{(A'_1(xA_1-A_2))^2+|A|^2}).$$

We note that $\varphi_1(\theta)$ and $\varphi_2(\theta)$ are the characteristic functions (c. f.' s) of $xA'_1W_MA_1-(A'_2W_MA_1+A'_1W_MA_2)/2$ and $xA'_1W_BA_1-(A'_2W_BA_1+A'_1W_BA_2)/2$ respectively. Moments of $F_j(x)$ can be obtained using the formula given in Evans and Savin (1981).

Corollary 1 : Let μ_{jk} be the k-th order central moment of $F_j(x)$. Then we have

$$\mu=\mu_{11}=\mu_{21}=A'_1A_2/A'_1A_1,$$
$$\sigma^2_1=\mu_{12}=k_1|A|^2/(A'_1A_1)^2,$$
$$\sigma^2_2=\mu_{22}=k_2|A|^2/(A'_1A_1)^2,$$
$$\mu_{13}=\mu_{23}=0,$$
$$\mu_{14}=l_1\sigma^4_1, \qquad \mu_{24}=l_2\sigma^4_2,$$

where

$$k_1=\frac{1}{4}\int_0^\infty u(\cosh u)^{-\frac{1}{2}}du-\frac{1}{2}=0.89072,$$

$$k_2=\frac{1}{12}\int_0^\infty u^{\frac{3}{2}}(\sinh u)^{-\frac{1}{2}}du-\frac{1}{2}=0.39652,$$

$$l_1=\left[\frac{7}{192}\int_0^\infty u^3(\cosh u)^{-\frac{1}{2}}du-k_1-\frac{1}{8}\right]/k^2_1 =4.95393,$$

$$l_2=\left[\frac{1}{320}\int_0^\infty u^{\frac{7}{2}}(\sinh u)^{-\frac{1}{2}}du-k_2-\frac{1}{8}\right]/k^2_2 =4.08381.$$

It is noticed that F_1 and F_2 have the same mean ; the variance of F_2 is smaller ; the kurtosis μ_{14}/σ^4_1-3 of F_1 is 1.95393 while that of F_2 is 1.08381. Since $\mu_{13}=\mu_{23}=0$, the skewness of F_1 and F_2 is 0. In fact F_1 and F_2 are symmetric about μ. To show this we first note from Appendix that

$$L(xA'_1W_MA_1-\frac{1}{2}(A'_2W_MA_1+A'_1W_MA_2)) =L\left(\sum_{n=1}^\infty\frac{\xi_1(x)X^2_n+\xi_2(x)Y^2_n}{\left(n-\frac{1}{2}\right)^2\pi^2}\right),$$

$$L(xA'_1W_BA_1-\frac{1}{2}(A'_2W_BA_1+A'_1W_BA_2)) =L\left(\sum_{n=1}^\infty\frac{\xi_1(x)X^2_n+\xi_2(x)Y^2_n}{n^2\pi^2}\right),$$

where $(X_n,\ Y_n)'\sim\mathrm{NID}(0,I_2)$. Then it can be checked easily that $F_j(x+\mu)+F_j(-x+\mu)=1$ $(j=1,2)$ implying symmetry about μ. Moreover it is found that $F_j(\sigma_jx+\mu)$ $(j=1,2)$ do not depend on A, i. e., the limiting distributions of $(\hat{B}-\mu)/\sigma_1$ and $(\tilde{B}-\mu)/\sigma_2$ are independent of A. More specifically we have the following corollary.

Corollary 2 : Let $G_1(x)$ and $G_2(x)$ be the limiting distribution functions of $(\hat{B}-\mu)/\sigma_1$ and $(\tilde{B}-\mu)/\sigma_2$ respectively. Then we have

$$G_j(x)=F_j(\sigma_jx+\mu) =\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\frac{1}{\theta}\mathrm{Im}(\varphi_j(\theta))\,d\theta,$$

where $\varphi_j(\theta)$ are the same as $\varphi_j(\theta)$ $(j=1,2)$ in Theorem 2 except that $\xi_1(x)$ and $\xi_2(x)$ are replaced by

$$\xi_1(x),\,\xi_2(x)=\frac{1}{2}(\sqrt{k_j}\,x\pm\sqrt{k_jx^2+1}), \qquad (j=1,2)$$

with k_1 and k_2 given in Corollary 1.

Figure 1 draws the graphs of $g_j(x)=dG_j(x)/dx$ together with the density of $N(0,1)$. All distributions have the mean 0, vari-

Figure 1 Probability Densities of $G_j(x)$ and $N(0, 1)$

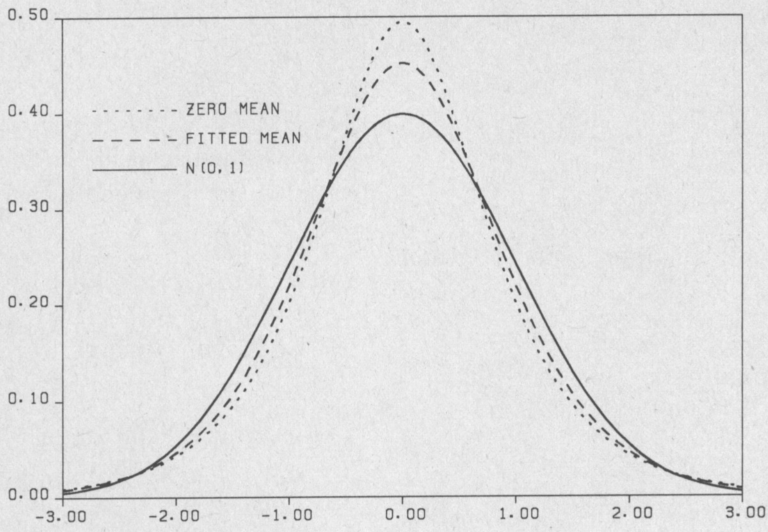

Table I Percent Points of $G_j(x)$

P	0.5	0.6	0.7	0.8	0.9	0.95	0.975	0.99
$x(j=1)$	0	0.20553	0.43324	0.72052	1.17607	1.61712	2.05470	2.63320
$x(j=2)$	0	0.22451	0.46927	0.76797	1.21674	1.62798	2.02047	2.52341

ance 1 and skewness 0. The kurtosis of G_1 is 1.95393 while that of G_2 is 1.08381, as was described before. Table 1 reports percent points of $G_j(x)$ for $x \geqq 0$. The percent point of $F_j(x)$ may be recovered from the percent point x_j of $G_j(x)$ as $\sigma_j x_j + \mu$. We note in passing that $G_2(x)$ corresponds to the limiting distribution function of the LSE in the spurious regression discussed in Granger and Newbold (1974) and Phillips (1986).

4. Singular Case-Cointegration

In Theorem 1 we observe that, if there exists a $q_2 \times q_1$ matrix B such that $A'_2 = BA'_1$, then $\hat{B}$ and $\tilde{B}$ both converge in probability to $B = A'_2 A_1 (A'_1 A_1)^{-1}$. In this case the matrix A in (2) becomes singular with $\text{rank}(A) = \text{rank}(A_1) = q_1$, which we assume in this section.

Let us rewrite (1) with (2) compactly as

$$(7) \quad (1-L)\begin{pmatrix} y_{1j} \\ y_{2j} \end{pmatrix} = A(L)\varepsilon_j = \begin{pmatrix} A'_1 \\ A'_2 \end{pmatrix}\varepsilon_j$$

$$+ (A(L) - A)\begin{pmatrix} \varepsilon_{1j} \\ \varepsilon_{2j} \end{pmatrix},$$

where L is the lag operator and

$$A(L) = \sum_{l=0}^{\infty} C_l L^l.$$

Then, premultiplying $(-B, I_{q2})$ on both sides of (7), we have

$$(8) \quad y_{2j} = By_{1j} + v_{2j},$$

where $\{v_{2j}\}$ is defined by

$$v_{2j} = G'(L)\varepsilon_j,$$
$$G'(L) = (-B, I_{q2})(A(L) - A)/(1-L): \quad q_2 \times q.$$

Note that

$$\frac{A(L) - A}{1-L} = -\sum_{l=1}^{\infty} C_l \left(\sum_{m=0}^{l-1} L^m \right),$$

which is well defined because of the second relation in (2) so that $\{v_{2j}\}$ becomes stationary. Note also that $(-B, I_{q2})$ is the cointegrating matrix which transforms $\{y_j\}$ into a stationary process $\{y_{2j}-By_{1j}\}$.

We now consider the asymptotic distributions of $\hat{B}$ and $\tilde{B}$ defined in (5) and (6) respectively. For this purpose we first have the following lemma, which is essentially due to Phillips and Durlauf (1986).

Lemma 2 : For the model (1) with (2) it holds that, as $T\to\infty$,

$$L\Big(\frac{1}{T}\sum_{j=1}^{T} y_j u'_j\Big)\to L\Big(A W_I A'+\sum_{j=0}^{\infty}\Gamma_j\Big),$$

$$L\Big(\frac{1}{T}\sum_{j=1}^{T}(y_j-\bar{y})u'_j\Big)$$
$$\to L\Big(A\Big(W_I-\int_0^1 w(t)\,dtw(1)'\Big)A'+\sum_{j=0}^{\infty}\Gamma_j\Big),$$

where

$$\bar{y}=\frac{1}{T}\sum_{j=1}^{T} y_j,\qquad \Gamma_j=E(u_k u'_{k+j}),$$
$$W_I=\int_0^{\infty} w(t)\,dw(t)'.$$

Let us put

$$\bar{A}=\begin{pmatrix} A'_1 \\ G' \end{pmatrix}\begin{matrix} \updownarrow\ q_1 \\ \updownarrow\ q_2 \end{matrix},$$

where

$$G'=G'(1)$$
$$=\Big(B\Big[\sum_{l=1}^{\infty} lC_l\Big]_{(1,1)}-\Big[\sum_{l=1}^{\infty} lC_l\Big]_{(2,1)},$$
$$B\Big[\sum_{l=1}^{\infty} lC_l\Big]_{(1,2)}-\Big[\sum_{l=1}^{\infty} lC_l\Big]_{(2,2)}\Big),$$

with $[M]_{(j,k)}$ being the $(j, k)-$th block of M. We also define

$$P=\begin{pmatrix} P'_1 \\ P'_2 \end{pmatrix}\begin{matrix} \updownarrow\ q_1 \\ \updownarrow\ q_2 \end{matrix},$$

which is a $q\times q$ lower triangular matrix such that $PP'=\bar{A}\bar{A}'$ with $\operatorname{rank}(P'_1)=q_1$.

Then we have the following theorem.

Theorem 3 : As $T\to\infty$, it holds that

$$L(T(\hat{B}-B))\to L((A'_1 W_I G+D)'(A'_1 W_M A_1)^{-1})$$
$$=L((P'_1 W_I P_2+D)'(P'_1 W_M P_1)^{-1}),$$
$$L(T(\tilde{B}-B))\to L\Big((A'_1\Big(W_I-\int_0^1 w(t)\,dt\, w(1)'\Big)G+D)'(A'_1 W_B A_1)^{-1}\Big)$$
$$=L\Big(P'_1\Big(W_I-\int_0^1 w(t)\,dt\, w(1)'\Big)P_2+D)'(P'_1 W_B P_1)^{-1}\Big),$$

where

$$D=\sum_{j=0}^{\infty}E(u_{1k}v'_{2,k+j})\ :\ q\times q_2.$$

Let us restrict to the case $q_1=q_2=1$ as in the previous section and denote the limiting distributions of $T(\hat{B}-B)$ and $T(\tilde{B}-B)$ as $H_1(x)$ and $H_2(x)$ respectively. We also put

$$P=\begin{pmatrix} \sqrt{A'_1A_1} & 0 \\ \dfrac{A'_1G}{\sqrt{A'_1A_1}} & \sqrt{G'G-\dfrac{(A'_1G)^2}{A'_1A_1}} \end{pmatrix}$$
$$=\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

Then we have the following results concerning the limiting distributions of $T(\hat{B}-B)$ and $T(\tilde{B}-B)$.

Theorem 4 :

$$H_j(x)=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\frac{1}{\theta}\operatorname{Im}(e^{i\theta(ab-2d)}\tilde{\varphi}_j(\theta))\,d\theta,\qquad (j=1, 2),$$

where $d=D$ and

$$\tilde{\varphi}_1(\theta)=\Big[\cos\sqrt{\nu}+2i\theta ab\frac{\sin\sqrt{\nu}}{\sqrt{\nu}}\Big]^{-\frac{1}{2}},$$
$$\tilde{\varphi}_2(\theta)=\Big[\frac{-8a^2b^2\theta^2}{\nu^2}(\cos\sqrt{\nu}-1)$$
$$+\Big(1-\frac{4a^2b^2\theta^2}{\nu}\Big)\frac{\sin\sqrt{\nu}}{\sqrt{\nu}}\Big]^{-\frac{1}{2}},$$
$$\nu=4i\theta a^2(x+i\theta c^2).$$

The first two moments of $H_j(x)$ are given by the following corollary.

Corollary 3 : Let μ'_{jk} be the k-th order raw moment of $H_j(x)$. Then we have

$$\mu'_{11}=\frac{2d-ab}{2a^2}m_1+\frac{b}{a},$$
$$\mu'_{21}=\frac{2d-ab}{2a^2}n_1,$$
$$\mu'_{12}=\frac{b^2}{2a^2}+\left(\frac{4c^2-3b^2}{4a^2}+\frac{2bd}{a^3}\right)m_1+\frac{(2d-ab)^2}{8a^4}m_2,$$
$$\mu'_{22}=\frac{b^2+2c^2}{2a^2}n_1+\frac{b^2}{a^2}n_2+\frac{(2d-ab)^2}{8a^4}n_3,$$

where

$$m_1=\int_0^\infty u(\cosh u)^{-\frac{1}{2}}du=5.56286,$$
$$m_2=\int_0^\infty u^3(\cosh u)^{-\frac{1}{2}}du=135.66249,$$
$$n_1=\int_0^\infty u^{\frac{3}{2}}(\sinh u)^{-\frac{1}{2}}du=10.75826,$$
$$n_2=\int_0^\infty u^{\frac{1}{2}}(\sinh u)^{-\frac{3}{2}}(1-\cosh u)\,du=-2.64149,$$
$$n_3=\int_0^\infty u^{\frac{7}{2}}(\sinh u)^{-\frac{1}{2}}du=372.35719.$$

In general $H_j(x)$ are not symmetric. There exists no transformation that makes the limiting distributions of $T(\hat{B}-B)$ and $T(\tilde{B}-B)$ independent of parameters a, b, c and d. It, however, can be shown that, if $b=d=0$, then $H_j(x)$ are symmetric about the origin. In that case $H_j(cx/a)$, i. e., the limiting distributions of $T(\hat{B}-B)$ and $T(\tilde{B}-B)$ multiplied by a/c, do not depend on a and c. We also have the following central moments of $H_j(cx/a)$ with $b=d=0$:

$$\mu=\mu'_{11}=\mu'_{21}=0,$$
$$\sigma^2{}_1=\mu'_{12}=m_1, \qquad \sigma^2{}_2=\mu'_{22}=n_1,$$
$$\mu_{13}=\mu_{23}=0,$$
$$\mu_{14}=\frac{3}{64}\int_0^\infty u^4(\cosh u)^{-\frac{1}{2}}\left(u(\tanh u)^2+\frac{2}{3}\tanh u-\frac{2}{3}u\right)du=203.49373,$$
$$\mu_{24}=\frac{3}{64}\int_0^\infty u^{\frac{7}{2}}(\sinh u)^{-\frac{1}{2}}\left(u^2(\coth u)^2-\coth u-\frac{2}{3}u^2\right)du=558.53578.$$

Therefore the kurtosis is 3.57589 for $j=1$ and 1.82578 for $j=2$.

Figure 2 draws graphs of the limiting probability densities of $T(\hat{B}-B)/\sigma_1\cdot a/c$ and $T(\tilde{B}-B)/\sigma_2\cdot a/c$, i. e., those of $H_j(c\sigma_j x/a)$ $(j=1, 2)$ for which $b=d=0$ together with the density of $N(0, 1)$. Table II reports percent points of $H_j(c\sigma_j x/a)$ with $b=d=0$.

(Department of Economics,
Hitotsubashi University)

Appendix

Proof of Theorem 2 : Let us consider first $F_1(x)=P(X_1>0)$, where

$$X_1=\int_0^1\int_0^1[1-\max(s, t)]dw(s)'Hdw(t),$$
$$H=xA_1A'_1-\frac{1}{2}(A_2A'_1+A_1A'_2).$$

Note that the eigenvalues of H are $\xi_1(x)$ and $\xi_2(x)$. It is known (see, for example, Varberg (1966)) that the eigenvalues of the integral equation

$$g(t)=\lambda\int_0^1[1-\max(s, t)]g(s)\,ds$$

are given by $\lambda=\left(n-\frac{1}{2}\right)^2\pi^2 (n=1, 2\cdots)$. Thus Mercer's theorem (Hochstadt (1973)) yields

$$L(X_1)=L\left(\int_0^1\int_0^1[1-\max(s, t)](\xi_1(x)dw_1(s)dw_1(t)+\xi_2(x)dw_2(s)dw_2(t))\right)$$
$$=L\left(\sum_{n=1}^\infty\frac{\xi_1(x)W^2{}_{1n}+\xi_2(x)W^2{}_{2n}}{\left(n-\frac{1}{2}\right)^2\pi^2}\right),$$

Figure 2 Probability Densities of $H_j(c\sigma_j x/a)$ and $N(0,1)$

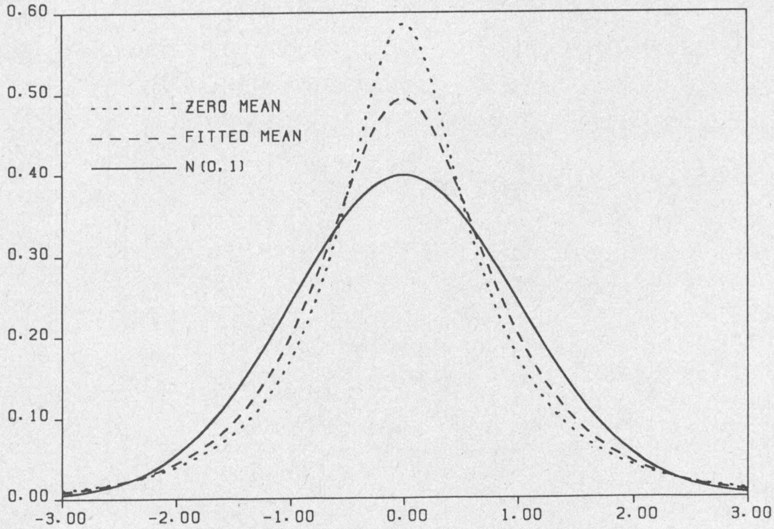

Table II Percent Points of $H_j(c\sigma_j x/a)$

P	0.5	0.6	0.7	0.8	0.9	0.95	0.975	0.99
$x(j=1)$	0	0.17437	0.37407	0.64288	1.11045	1.59910	2.10493	2.79217
$x(j=2)$	0	0.20569	0.43422	0.72379	1.18434	1.63078	2.07748	2.69403

where $(W_{1n}, W_{2n})' \sim NID(0, I_2)$. Then it can be checked that the c. f. $\varphi_1(\theta)$ of X_1 is given as in the theorem, which leads us to the expression for $F_1(x)$.

The expression for $F_2(x)=P(X_2>0)$ can be proved similarly, where

$$X_2=\int_0^1\int_0^1[\min(s, t)-st]dw(s)'Hdw(t).$$

Since the eigenvalues for the integral equation with the kernel $\min(s, t)-st$ are given by $n^2\pi^2(n=1, 2, \cdots)$, we arrive at the c. f. $\varphi_2(\theta)$ of X_2 as given in the theorem to obtain $F_2(x)$.

Proof of Corollary 1: Let us denote by μ'_{jk} the k-th order raw moment of $F_j(x)$. Then, because of the formula given in Evans and Savin (1981), we have

$$\mu'_{jk}=\frac{1}{\Gamma(k)}\int_0^\infty \theta_1^{k-1}\left(\frac{\partial^k}{\partial\theta^k_2}m_j(-\theta_1, \theta_2)\right)\bigg|_{\theta_2=0} d\theta_1,$$

where

$$m_1(-\theta_1, \theta_2)=[\cos\sqrt{a_1+a_2}\cos\sqrt{a_1-a_2}]^{-\frac{1}{2}},$$

$$m_2(-\theta_1, \theta_2)=\left[\frac{\sin\sqrt{a_1+a_2}}{\sqrt{a_1+a_2}}\frac{\sin\sqrt{a_1-a_2}}{\sqrt{a_1-a_2}}\right]^{-\frac{1}{2}},$$

$$a_1=-\theta_1A'_1A_1+\theta_2A'_1A_2,$$

$$a_2=a^2{}_1+\theta^2{}_2|A|^2.$$

Partially differentiating $m_j(-\theta_1, \theta_2)$ with respect to θ_2 and evaluating at $\theta_2=0$ we obtain results described in the corollary. In the course of computation we used computer algebra REDUCE.

Proof of Theorem 4: We have $H_j(x)=P(Y_j>0)$ $(j=1, 2)$, where

$$Y_1=a^2x\int_0^1\int_0^1[1-\max(s, t)]dw_1(s)dw_1(t)-P'_1W_IP_2-d,$$

$$Y_2=a^2x\int_0^1\int_0^1[\min(s, t)-st]dw_1(s)dw_1(t)-P'_1(W_I-\int_0^1 w(t)dw(t)')P_2-d,$$

$$P'_1=(a, 0), \qquad P'_2=(b, c).$$

Suppose that $\varepsilon_j=(\varepsilon_{1j}, \varepsilon_{2j})'\sim\text{NID}(0, I_2)$ and define

$$C=\begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}: T\times T, \qquad e=\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}: T\times 1,$$

$$M=I_T-\frac{1}{T}ee', \qquad f=(T,\ T-1, \cdots, 1)',$$

$$\varepsilon_1=(\varepsilon_{11}, \cdots, \varepsilon_{1T})', \qquad \varepsilon_2=(\varepsilon_{21}, \cdots, \varepsilon_{2T})',$$

$$\varepsilon=(\varepsilon'_1, \varepsilon'_2)'.$$

Then it can be shown using Lemmas 1 and 2 that $L(Y_{1T})\to L(2Y_1)$ and $L(Y_{2T})\to L(2Y_2)$, where

$$Y_{1T}=\varepsilon'\begin{pmatrix} \frac{2a^2x}{T^2}C'C-\frac{ab}{T}ee' & -\frac{ac}{T}C' \\ -\frac{ac}{T}C & \mathrm{O} \end{pmatrix}\varepsilon$$

$$+ab-2d,$$

$$Y_{2T}=$$

$$\varepsilon'\begin{pmatrix} \frac{2a^2x}{T^2}C'MC-\frac{ab}{T}\left(ee'-\frac{1}{T}fe'-\frac{1}{T}ef'\right) & -\frac{ac}{T}C'M \\ -\frac{ac}{T}MC & 0 \end{pmatrix}\varepsilon+ab-2d.$$

Thus we have only to obtain the limiting c. f.'s of Y_{1T} and Y_{2T}.

Let the c. f.'s of Y_{1T} and Y_{2T} be $\varphi_{1T}(\theta)$ and $\varphi_{2T}(\theta)$ respectively. Then we have

$$\varphi_{iT}(\theta)=\left|I_T-\frac{2i\theta}{T}B_{jT}\right|^{-\frac{1}{2}}e^{i\theta(ab-2d)}, \qquad (j=1, 2),$$

where

$$B_{1T}=\frac{\delta}{T}C'C-abee',$$

$$B_{2T}=\frac{\delta}{T}C'MC-ab\left(ee'-\frac{1}{T}fe'-\frac{1}{T}ef'\right),$$

$$\delta=2a^2(x+i\theta c^2).$$

For these B_{jT}'s there exist continuous and symmetric functions $K_j(s, t)$ defined on $[0, 1]\times[0, 1]$ that satisfy

$$\lim_{T\to\infty}\max_{k,l}\left|B_{jT}(k, l)-K_j\left(\frac{k}{T}, \frac{l}{T}\right)\right|=0,$$

where

$$K_1(s, t)=\delta(1-\max(s, t))-ab,$$

$$K_2(s, t)=\delta(\min(s, t)-st)-ab(s+t-1).$$

It is impossible to obtain eigenvalues of K_1 and K_2 explicitly, but we can derive the Fredholm determinants $D_j(\lambda)$ associated with K_j as follows.

$$D_1(\lambda)=\cos\sqrt{\lambda\delta}+\lambda ab\frac{\sin\sqrt{\lambda\delta}}{\sqrt{\lambda\delta}},$$

$$D_2(\lambda)=\frac{2a^2b^2}{\delta^2}(\cos\sqrt{\lambda\delta}-1)+\left(1+\frac{\lambda a^2b^2}{\delta}\right)\frac{\sin\sqrt{\lambda\delta}}{\sqrt{\lambda\delta}}.$$

Then $\varphi_{jT}(\theta)\to[D_j(2i\theta)]^{-1/2}\exp(i\theta(ab-2d))$ as $T\to\infty$, which gives $\tilde{\varphi}_j(\theta)$ in the theorem. This, in turn, gives expressions for $H_j(x)$.

References

[1] Engle, R. F. and C. W. J. Granger, "Co-integration and Error Correction : Representation, Estimation, and Testing," *Econometrica*, 55(1987), pp. 251-276.

[2] Engle, R. F. and B. S. Yoo, "Forecasting and Testing in Co-integrated Systems," *Journal of Econometrics*, 35(1987), pp. 143-159.

[3] Evans, G. B. A. and N. E. Savin, "Testing for Unit Roots : 1," *Econometrica*, 49(1981), pp. 753-779.

[4] Granger, C. W. J., "Some Properties of Time Series Data and Their Use in Econometric Model Specification," *Journal of Econometrics*, 16(1981), pp. 121-130.

[5] Granger, C. W. J. and P. Newbold, "Spurious Regressions in Econometrics," *Journal of Econometrics*, 2(1974), pp. 111-120.

[6] Hochstadt, H., *Integral Equations*, New York : John Wiley, 1973.

[7] Phillips, P. C. B., "Understanding Spurious Regressions in Econometrics," *Journal of Econometrics*, 33(1986), pp. 311-340.

[8] Phillips, P. C. B. and S. N. Durlauf, "Multiple Time Series Regression with Integrated Processes," *Review of Economic Studies*, 53(1986), pp. 473-495.

[9] Phillips, P. C. B. and S. Ouliaris, "Asymptotic Properties of Residual Based Tests for Cointegration," *Cowles Foundation Discussion Paper No. 847* (1987).

[10] Tanaka, K., "The Fredholm Approach to Asymptotic Inference on Nonstationary and Noninvertible Time Series Models," (To appear in) *Econometric Theory*, 6(1990).

[11] Varberg, D. E., "Convergence of Quadratic Forms in Independent Random Variables," *Annals of Mathematical Statistics*, 37(1966), pp. 567-576.