The Optimum Interest Rate under Uncertain Life Time*

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1. Introduction

The famous paper of Diamond (1965) analyzed the growing decentralized economy with the neoclassical production function, and obtained the following result. The steady state equilibrium interest rate is not necessarily equal to the population growth rate, and the government can control the interest rate by the national debt policy. Stein (1969), Samuelson (1975), and Ihori (1978) discussed the problem of the realization of the gold en rule steady state by the national debt policy or the social security system. In these papers it is assumed that all people in the economy live their whole life time. In this case the optimality of the golden rule interest rate is easily proved. But if we take into consideration the uncertainty of life time, the golden rule interest rate, which is equal to the population growth rate, is not optimum. The life time uncertainty has been considered in the problem of the individual life time consumption by Yaari (1965) and Levhari and Mirman (1977). In this paper we will consider its effect on the equilibrium interest rate.

The purpose of this paper is to show the following results. 1) The optimum interest rate is greater than the population growth rate in the economy with uncertainty of life time. 2) The optimum steady state can be realized by the government's national debt policy. It is the generalization of the previous result presented by Diamond, Stein, Samuelson and Ihori. The quantity of the national debt which realize the optimum steady state in the economy with uncertainty of life time is greater than that in the economy without uncertainty of life time

2. The model and the individual behaviour

We suppose that the economy consists of the two overlapping generations. They may live two periods, but they are not necessarily alive in the second period. We denote the probability that the individual is alive in the second period as p. It is common to all individuals and $0<p<1$. They work in the first period and retire in the second period. They are *ex ante* identical, have the same preference and ability. They save a part of the income obtained in the first period and, if alive, consume the savings in the second period. The savings are invested in capital and national debt which are the perfect substitutes. The savings of the individuals who die halfway are left as the "unintended bequests." We assume that this bequests are collected by the government and distributed equally to the younger generation. We call it as "inheritance."

The individuals maximize the following expected utility.

$$Eu=u(c_1)+p\beta u(c_2) \tag{1}$$

u is the von Neuman-Morgenstern utility function and is strictly concave. c_1 and c_2 are the consumptions in the first and the second period. β is the discount factor and $0<\beta<1$.

The budget constraint for the individuals who are born in the t period is

$$c_1+\frac{1}{1+r_{t+1}}c_2=w_t+v_t+h_t=y_t \tag{2}$$

r_{t+1} is the interest rate in the $t+1$ period. y_t is the life time income of the t generation. w_t is the wage rate, v_t is the lump sum transfer inome and h_t is the inheritance defined above, in the t period. If v_t is negative, it is regarded as the lump sum tax. The savings of t generation is represented as

$$s_t=y_t-c_1=\frac{1}{1+r_{t+1}}c_2 \tag{3}$$

v_t and h_t are determined by the following equations.

$$v_t=\frac{n-r_t}{1+n}g \tag{4}$$

$$h_t=(1-p)\frac{1+r_t}{1+n}s_{t-1} \tag{5}$$

n is the population growth rate. g is the per capita national debt. We assume that g is constant, because it is determined by the government's policy. If it is negative, it is regarded as capital owned by the government. s_{t-1} is the $t-1$ gener-

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ation's savings. h_t is equal to the unintended bequests of $t-1$ generation per t generation individual. It is predetermined for the t generation.

The first order condition of expected utility maximization subject to (2) is

$$E_1=(1+r_{t+1})E_2 \tag{6}$$

where $E_1=\frac{\partial E}{\partial c_1}=\frac{\partial u}{\partial c_1}$ and $E_2=\frac{\partial E}{\partial c_2}=p\beta\frac{\partial u}{\partial c_2}$

The second order condition is

$$E_{11}+(1+r_{t+1})^2E_{22}<0 \tag{7}$$

where $E_{11}=\frac{\partial^2 E}{\partial {c_1}^2}$ and $E_{22}=\frac{\partial^2 E}{\partial {c_2}^2}$

It is satisfied by the concavity of u. From (2) and (6), we obtain

$$0<s_y=\frac{\partial s_t}{\partial y_t}=\frac{E_{11}}{E_{11}+(1+r_{t+1})^2E_{22}}<1 \tag{8}$$

$$s_r=\frac{\partial s_t}{\partial r_{t+1}}=-\frac{E_2+(1+r_{t+1})s_tE_{22}}{E_{11}+(1+r_{t+1})^2E_{22}} \tag{9}$$

$$s_p=\frac{\partial s_t}{\partial p}=-\frac{1+r_{t+1}}{p}\frac{E_2}{E_{11}+(1+r_{t+1})^2E_{22}}>0 \tag{10}$$

(10) implies that the increase in the uncertainty of life time (decrease in p) reduces the individual savings. It is the simplified and stronger version of the result presented by Levhari and Mirman (1977). It includes the effect of shortening of the horizon as well as that of uncertainty of the horizon.

3. Market equilibrium condition

Capital market equilibrium condition is

$$(1+n)k_{t+1}+g=s_t \tag{11}$$

In the steady state

$$(1+n)k+g=s \tag{12}$$

Where k_t and k are per capita capital (or capital intensity). We assume the neoclassical production function which has constant returns to scale and diminishing marginal returns properties. Denoting the output per labour as $f(k)$, we obtain

$$r=f'(k)$$

$$w=f(k)-rf'(k),\quad \frac{dw}{dr}=-k$$

and from diminishing marginal returns, $f''(k)<0$.

4. The consumption maximizing interest rate

In the steady state the individual budget constraint is

$$c_1+\frac{1}{1+r}c_2=w+v+h=f-kr+\frac{n-r}{1+n}g+(1-p)\frac{1+r}{1+n}s \tag{13}$$

and

$$s=\frac{1}{1+r}c_2 \tag{14}$$

Substituting (12) into (13), we obtain

$$c_1+\frac{1}{1+r}c_2=f-kr+\frac{n-r}{1+n}[s-(1+n)k]+(1-p)\frac{1+r}{1+n}s$$
$$=f-nk+\frac{1}{1+r}c_2-\frac{p}{1+n}c_2$$

From this,

$$c_1+\frac{p}{1+n}c_2=f-nk \tag{15}$$

where nk is the steady state investment. $\frac{p}{1+n}$ is the population ratio of the older generation alive in the second period to the younger generation. Denoting the population of the younger generation as N_t, the total consumption in the t period is, from (15)

$$c_1N_t+\frac{p}{1+n}N_tc_2=(f-nk)N_t \tag{16}$$

Differentiating (16) with respect to k, we obtain the following first order condition for maximization of the total consumption.

$$f'(k)=r=n \tag{17}$$

The second order condition is satisfied by concavity of $f(k)$. Uncertainty of life time does not affect the per labour output, $f(k)$, and the steady state investment, nk. Therefore, the condition of consumption maximization under uncertainty of life time is the same as traditional golden rule.

5. Stability condition and comparative statics

From (12) the steady state capital intensity, k, and the interest rate, r, are determined by g.

According to Diamond (1965) we consider the stability condition of the steady state equilibrium and comparative statics problem. In the beginning we assume that the capital market is Walras stable, in other words, the excess demand for capital is increasing with respect to the interest rate. This condition is represented as

$$s_r-(1+n)\frac{1}{f''}>0$$

From $f''<0$

$$1+n-s_rf''>0 \tag{18}$$

Next, from (11)

$$s_{t-1}=(1+n)k_t+g$$

Substituting it into (5)

$$h_t = (1-p)(1+r_t)\left(k_t+\frac{g}{1+n}\right) \tag{19}$$

Differentiating (4) and (19) with respect to r_t,

$$\frac{dv_t}{dr_t} = -\frac{g}{1+n} \tag{20}$$

$$\frac{dh_t}{dr_t} = (1-p)(1+r_t)\frac{1}{f''} + (1-p)\left(k_t+\frac{g}{1+n}\right) \tag{21}$$

From these equations,

$$\frac{ds_t}{dr_t} = s_y\left(\frac{dw_t}{dr_t}+\frac{dv_t}{dr_t}+\frac{dh_t}{dr_t}\right) = s_y\left[(1-p)(1+r_t)\frac{1}{f''} - p\left(k_t+\frac{g}{1+n}\right)\right] \tag{22}$$

The stability condition of the steady state equilibrium is obtained by differentiating (11) with respect to r_t as follows

$$0 < \frac{dr_{t+1}}{dr_t} = \frac{ds_t}{dr_t}\frac{f''}{1+n-s_r f''} = \frac{s_y\left[(1-p)(1+r_t) - p\left(k_t+\frac{g}{1+n}\right)f''\right]}{1+n-s_r f''} < 1 \tag{23}$$

The comparative statics result of the steady state equilibrium interest rate is obtained by differentiating (12) with respect to g.[1)]

$$\frac{dr}{dg} = \frac{\left[s_y - 1 - \frac{1+r}{1+n}ps_y\right]f''}{1+n-s_r f'' - s_y\left[(1-p)(1+r) - p\left(k+\frac{g}{1+n}\right)f''\right]} > 0 \tag{24}$$

The denominator of the right hand side of (24) is positive from (18) and (23), and the numerator is also positive from (8). From (24) we can say that with the assumption of the Walras stability of the capital market and the stability of the steady state equilibrium the steady state interest rate can be controlled by the national debt policy of the government.

In the next section we consider the optimum national debt policy or the optimum interest rate under uncertainty of life time. We will show that the optimum interest rate is greater than the population growth rate if $p<1$.

6. The optimum interest rate

In this section we consider the condition for maximization of the steady state individual expected utility.

From (10) in section 2 we obtain the result that the individual savings under uncertain life time is smaller than that under certain life time. Therefore we have conjecture that the optimum interest rate under which the steady state individual expected utility is maximized is larger than the population growth rate, although the consumption maximizing interest rate is equal to the population growth rate. We shall prove the following proposition.

Proposition

"The optimum interest rate under which the steady state individual expected utility is maximized is greater than the population growth rate, n, and less than $\frac{1+n}{p}-1$, if $p<1$."

Proof

"Differentiating $E(u)$ in (1) with respect to g, we obtain the following condition for maximization of expected utility.

$$\frac{dE}{dg} = E_1\frac{dc_1}{dg} + E_2\frac{dc_2}{dg} = (1+r)E_2\left(\frac{dc_1}{dg}+\frac{dc_2}{1+rdg}\right) = 0 \tag{25}$$

The second equality is obtained from (6).

From (12) and (15)

$$c_1 = f - nk - p(1+r)k - \frac{1+r}{1+n}pg \tag{26}$$

$$c_2 = (1+n)(1+r)k + (1+r)g \tag{27}$$

These two equations represent the combination of the steady state consumptions.

Differentiating (26) and (27) with respect to g,

$$\frac{dc_1}{dg} = \left\{[r-n-p(1+r)]\frac{1}{f''} - p\left(k+\frac{g}{1+n}\right)\right\}\frac{dr}{dg} - \frac{1+r}{1+n}p \tag{28}$$

$$\frac{dc_2}{dg} = (1+n)\left[(1+r)\frac{1}{f''}+k+\frac{g}{1+n}\right]\frac{dr}{dg} + 1 + r \tag{29}$$

Substituting (24) into (28) and (29),[2)]

$$\frac{dc_1}{dg} = \frac{1}{T}\left[(1-s_y)(n-r) + (1-s_y)\frac{s}{1+n}pf'' + \frac{1+r}{1+n}ps_r f''\right] \tag{30}$$

$$\frac{dc_2}{dg} = \frac{1}{T}[(1+r)(n-r)s_y - (1-s_y)sf'' - s_r(1+r)f''] \tag{31}$$

where T is the denominator of (24) which is positive. Next, substituting (8) and (9) into (30)

1) Derivation of (24) is presented in appendix (1).

2) Derivations of (30), (31), (32) and (33) are presented in appendix (2).

and (31),

$$\frac{dc_1}{dg}=\frac{1}{TD}\left[(n-r)(1+r)^2E_{22}-\frac{1+r}{1+n}pE_2f''\right] \tag{32}$$

$$\frac{dc_2}{dg}=\frac{1}{TD}[(n-r)(1+r)E_{11}+(1+r)E_2f''] \tag{33}$$

where (D) is the denominator of (8) and (9) which is negative. Finally substituting (32) and (33) into (25), we obtain

$$\frac{dE}{dg}=\frac{(1+r)E_2}{T}\left[(n-r)+f''\left(1-\frac{1+r}{1+n}p\right)\frac{E_2}{D}\right]=0 \tag{34}$$

From (34), when $r\leqq n$, $\frac{dE}{dg}>0$, and when $r\geqq\frac{1+n}{p}-1$, $\frac{dE}{dg}<0$. Therefore we conclude that the optimum interest rate is greater than n and less than $\frac{1+n}{p}-1$. (Q. E. D.)"

If $p=1$, (34) implies that the optimum interest rate is equal to n. Thus this proposition is the extension of traditional golden rule.

Consider above conclusion in terms of marginal rate of substitution (MRS) and marginal rate of transformation (MRT). At the equilibrium MRS of c_1 to c_2 is obtained from (6) as

$$\text{MRS}=1+r$$

On the other hand MRT of c_1 to c_2 is obtained from (28) and (29) as

$$\text{MRT}=-\left(\frac{dc_2}{dg}\right)\Big/\left(\frac{dc_1}{dg}\right)$$

The condition for maximization of expected utility, (25), implies that MRS=MRT at the optimum interest rate.

When $r=n$, MRT is equal to $\frac{1+n}{p}$, the population ratio of the younger to the older alive in the second period. On the other hand MRS is equal to $1+n$ at $r=n$, because the individual budget constraint is not affected by uncertainty of life time besides augmentation of the income by the inheritance. Therefore MRS is not equal to MRT at $r=n$. Since, from (32) and (33), $\frac{dc_1}{dg}<0$ and $\frac{dc_2}{dg}>0$ at $r=n$, and $\frac{dr}{dg}>0$ from (24), we can realize the optimum growth path at the interest rate greater than n by increasing g to decrease c_1 and increase c_2, then to reduce MRT and increase MRS.

On the other hand, when $1+r=\frac{1+n}{p}$ or $r=\frac{1+n}{p}-1$, MRS is equal to $\frac{1+n}{p}$ but MRT is not. We think that, if total consumption is not affected by k or r, $1+r=\frac{1+n}{p}$ is optimum. However, as r deviates from n, total consumption decreases. So $1+r=\frac{1+n}{p}$ is not optimum.

7. Appendix

(1) To derive (24), differentiating (12) with respect to g,

$$(1+n)\frac{1}{f''}\frac{dr}{dg}+1=s_r\frac{dr}{dg}+s_y\left(\frac{dv}{dr}\frac{dr}{dg}+\frac{dh}{dr}\frac{dr}{dg}+\frac{\partial v}{\partial g}+\frac{\partial h}{\partial g}+\frac{dw}{dr}\frac{dr}{dg}\right)$$

From this

$$\frac{dr}{dg}=\frac{f''\left[s_y\left(\frac{\partial v}{\partial g}+\frac{\partial h}{\partial g}\right)-1\right]}{1+n-s_rf''-s_y\left(\frac{dv}{dr}+\frac{dh}{dr}+\frac{dw}{dr}\right)f''} \tag{A-1}$$

And

$$\frac{\partial v}{\partial g}+\frac{\partial h}{\partial g}=\frac{n-r}{1+n}+(1-p)\frac{1+r}{1+n}=1-\frac{1+r}{1+n}p \tag{A-2}$$

Substituting (A-2) and (22) into (A-1), we obtain (24).

(2) To derive (32) and (33), substituting (24) into (28) and (29),

$$\begin{aligned}\frac{dc_1}{dg}&=\frac{1}{T}\left(s_y-1-\frac{1+r}{1+n}ps_y\right)f''\left[(r-n-p-pr)\frac{1}{f''}-\frac{ps}{1+n}\right]-\frac{1}{T}\frac{1+r}{1+n}p\left[1+n-s_rf''-s_y(1-p)(1+r)+s_y\frac{ps}{1+n}f''\right]\\&=\frac{1}{T}\left\{[1+n-s_y(1-p)(1+r)]\left(-\frac{1+r}{1+n}p\right)+(r-n-p-pr)\left(s_y-1-\frac{1+r}{1+n}ps_y\right)\right\}\\&\quad+\frac{1}{T}f''\left[-\frac{ps}{1+n}\left(s_y-1-\frac{1+r}{1+n}ps_y\right)-\frac{1+r}{1+n}p\left(-s_r+s_y\frac{ps}{1+n}\right)\right]\\&=\frac{1}{T}\left[(1-s_y)(n-r)+(1-s_y)\frac{s}{1+n}pf''+\frac{1+r}{1+n}ps_rf''\right]\end{aligned} \tag{30}$$

$$\frac{dc_2}{dg}=\frac{1}{T}\left(s_y-1-\frac{1+r}{1+n}ps_y\right)f''(1+n)\left[(1+r)\frac{1}{f''}+\frac{s}{1+n}\right]+\frac{1}{T}(1+r)\left[1+n-s_rf''-s_y(1-p)\times(1+r)+s_y\frac{ps}{1+n}f''\right]$$

$$=\frac{1}{T}\left\{[1+n-s_y(1-p)(1+r)](1+r)+(1+r)\times\left(s_y-1-\frac{1+r}{1+n}ps_y\right)(1+n)\right\}+\frac{1}{T}f''\left[s\left(s_y-1-\frac{1+r}{1+n}ps_y\right)+(1+r)\left(-s_r+s_y\frac{ps}{1+n}\right)\right]$$

$$=\frac{1}{T}[(1+r)(n-r)s_y-(1-s_y)sf''-s_r(1+r)f''] \qquad (31)$$

Substituting (8) and (9) into (30) and (31), we obtain (23) and (33) as follows.

$$\frac{dc_1}{dg}=\frac{1}{TD}\left\{(1+r)^2E_{22}(n-r)+(1+r)^2E_{22}\times\frac{s}{1+n}pf''-\frac{1+r}{1+n}p[E_2+(1+r)sE_{22}]f''\right\}$$

$$=\frac{1}{TD}\left[(n-r)(1+r)^2E_{22}-\frac{1+r}{1+n}pE_2f''\right] \qquad (32)$$

$$\frac{dc_2}{dg}=\frac{1}{TD}\{(1+r)(n-r)E_{11}-(1+r)E_{22}sf''+[E_2+(1+r)sE_{22}](1+r)f''\}$$

$$=\frac{1}{TD}[(n-r)(1+r)E_{11}+(1+r)E_2f''] \qquad (33)$$

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