Equity and Fairness in an Economy with Public Goods*

Toshihiro Sato

1. Introduction

Pareto efficiency, which is rather weak and widely accepted by most economists as an efficiency criterion, does not help us in finding a way to cut a cake among individuals in an unambiguous way, since it does not involve any distributive consideration. In recent years, many studies have been devoted to setting up distributive criteria called *equity*. These studies may be classified into two groups according to their treatment of the initial distribution of goods among individuals. The basic point of view common to the criteria of the first group is that the initial distribution does not deserve to be regarded as just so that it is necessary to set up equity criteria in such a way as to eliminate any reference to the initial distribution. The equity concepts belonging to this group are the concepts of nonenviness proposed by Foley [6], egalitarian-equivalence by Pazner and Schmeidler [16], and average-envy-free allocation by Thomson [18]. The equity criterion that constitutes the second group is the concept of equitable net trade proposed by Schmeidler and Vind [17]: a net trade is said to be equitable if no one prefers anyone else's net trade to his own. The underlying point of view of this concept is that the initial distribution of goods should be regarded as the basis for distribution.

These distributive criteria have been created to be applied to allocations of private goods, whereas no distributive consideration has yet been made for allocations of public goods. This paper aims to set up a new equity concept for allocations of public goods along the equitable net trade concept.

A straightforward extension to the public good cases of the concept of fair net trade for private goods restricted to the case of one public and one private good leads to the conclusion that the equitable way to apportion the costs of producing a certain amount of public good among individuals is such that each individual shares the equal amount of costs. Further, I think that, even if we consider an economy with more than one public and more than one private good, the conclusion that each individual should bear the same amount of costs expressed in terms of a certain standard of measure would not be altered drastically. However, it should be noted that the equal cost sharing may be accepted as a distributive justice only if each individual is likely to receive the same amount of benefits from the public goods. This may be so with some kind of public goods such as national defence. But this may not always be true considering that the benefits derived from some kinds of public goods are different among individuals. To illustrate more precisely the nature of public goods I have in mind, I will present an example in the next section. In any case, it is indispensable to contemplate the factors causing individuals to derive

* The author is greatly indebted to Professor Mikiro Otsuki for his helpful comments and his encouragement in writing this paper. The comments of the referees are also gratefully acknowledged.

different amounts of benefits from the public goods. In the sequel, therefore, we will construct a model and propose a new equity concept for public good allocations in such a way as to treat the differences in the amounts of benefits explicitly.

To say in more detail, we introduce into the model as data the different characteristics of individuals which will explain the differences of the benefits individuals derive from the public goods. It should be noted that these characteristics are not the preferences nor the initial endowments of goods but other observable factors which bring about the differences in the enjoyment of benefits. The concept of consumer's surplus is utilized for measuring the amounts of benefits accruing to individuals and for defining equity and fairness, which may be called *equity in terms of consumer's surplus* (ECS) and *fairness in terms of consumer's surplus* (FCS), respectively. We will show in this paper that there exists a FCS allocation, and that it is unique and individually rational.

The rest of this paper is organized as follows. As has been already mentioned, section 2 is devoted to illustrating the nature of public goods to be considered throughout this paper. In section 3, we describe the basic structure of the economy. The definitions of ECS and FCS allocations will be formulated in section 4. Moreover, the existence of a FCS allocation is verified in this section. In section 5, we propose a continuous time planning procedure for attaining the FCS allocation, which is a slight modification of the MDP procedure proposed by Drèze and de la Vallée Poussin [5] and Malinvaud ([12] and [13]). The proofs of all theorems will be relegated to the last section.

2. Public Goods and Benefits

Throughout this paper we consider an economy with public goods where individuals derive different amounts of benefits from such public goods. In order to illustrate the nature of public goods, I present the following example:

Example. Suppose that it is planned to build a new road whose route is already decided by geographical reasons, and that individuals differ only in the ability to use this road. For instance, Mr. A possesses two cars while Mr. B possesses only one car. Then no one would regard an allocation as equitable in which Mr. A bears the same amount of costs of building this road as Mr. B does. The apportionment of costs has to be done by taking consideration of the difference in individual factors characterizing individual ability to use this road, say the number of cars they possess. ||

There may be many other cases where the observable characteristics of individuals are crucial factors in deciding whether an assignment of costs is equitable or not. In such cases, it would be necessary to build these observable characteristics into the model explicitly in considering distributive justice. Hereafter, we will construct a model faithfully to the idea represented by the above example.

3. The Model

Let us consider an economy with n individuals (or consumers), indexed by $i \in N := \{1, 2, \ldots, n\}$, producers and the planning bureau. Let there also be one public and one private good whose quantities are represented by x and y, respectively.

The planning bureau is charged with revising the allocation of resources.

Since we will concentrate on the equity concept in this economy, we need not specify in all the details about the producers. It is sufficient to consider the social production set which describes the possibilities for producing the public good from the private good. Here we assume that the planning bureau has the precise information as to this social production technology.

As has been already mentioned, we consider the observable characteristics of individuals which bring about the difference in the amounts of benefits from public goods as an important factor. Here notice that these observable characteristics of each individual are not his preferences nor his initial endowments of goods but his specific characteristics which affect the benefits he derives from the public good, say the number of his cars. Let $(\theta_1, \theta_2, \cdots, \theta_n)$ be a profile of parameters where θ_i represents the observable characteristics of individual i. Then each individual i is characterized by $\{u^i(\cdot, \cdot), w^i, \theta_i\}$ where $u^i(\cdot, \theta_i)$ is i's utility function defined on his consumption set R_+^2, and $w^i\,(>0)$ describes his endowment of private good. For simplicity, we assume that the individuals differ only in their characteristics: i. e., we assume $u^i(\cdot, \cdot) = u(\cdot, \cdot)$ for all i.

Now we introduce a series of assumptions and some familiar concepts that we will retain throughout this paper.

Assumption 1. The social production set is denoted by a cost function $f: R_+ \rightarrow R_+$ which is at least twice continuously differentiable and satisfies

$f(0) = 0,\ f'(x) := \gamma(x) > 0$ and $f''(x) = \gamma'(x) \geqq 0$ for all $x \geqq 0$.

Here the costs are measured in terms of the private good.

Assumption 2. For all i, the utility function $u(\cdot, \theta_i)$, given his characteristics θ_i, is strictly quasi-concave in the interior of R_+^2, at least twice continuously differentiable, and satisfies

$$u_x(x, y, \theta_i) := \frac{\partial}{\partial x} u(x, y, \theta_i) \geqq 0,$$

$$u_y(x, y, \theta_i) := \frac{\partial}{\partial y} u(x, y, \theta_i) > 0,$$

$$u_x(x, 0, \theta_i) = 0$$

for all $(x, y) \geqq 0$. Here, at the boundary of the domain, these partial derivatives are defined as the right hand side ones. Therefore, the marginal rate of substitution between the public and the private good, denoted by $\pi(x, y, \theta_i) := u_x(x, y, \theta_i) / u_y(x, y, \theta_i)$, is nonnegative and continuous in its first two arguments. Moreover, I make the following assumption.

Assumption 3. Let $w = \sum_i w^i$. Then for any i, given his characteristics θ_i, there exists some $y < +\infty$ such that

$u(0, y, \theta_i) \geqq u(f^{-1}(w), w, \theta_i)$.

Hereafter $\sum_i$ signifies the summation running over N. This last assumption implies that the public good considered is not a subsistence good, so that every individual with his consumption bundle (x, y), which is technically producible, can be always compensated only by the private good.

Throughout this paper, we make use of the following concepts. An allocation is an $(n+1)$-tuple of real numbers $(x, y^1, \cdots, y^n)$ where y^i denotes the quantity of private good allotted to individual i. An allocation $\boldsymbol{z} = (x, y^1, \cdots, y^n)$ constitutes a feasible allocation if

(1) $(x, y^1, \cdots, y^n) \geqq 0$ and $f(x) + \sum_i y^i \leqq \sum_i w^i$.

Let Z be the set of all feasible allocations. Moreover, let $\bar{Z}$ be the set of feasible allocations which lie on the production frontier. That is, an allocation $(x, y^1, \cdots, y^n)$ belongs to $\bar{Z}$ if $(x, y^1, \cdots, y^n) \geqq 0$ and $f(x) + \sum_i y^i = \sum_i w^i$. An allocation is said to be individually rational if each individual prefers it to his own initial holdings of goods. A Pareto efficient allocation for this economy is feasible allocation $\boldsymbol{z} = (x, y^1, \cdots, y^n)$ such that there is no other feasible allocation $\boldsymbol{z} = (\bar{x}, \bar{y}^1, \cdots, \bar{y}^n)$ satisfying $u(\bar{x}, \bar{y}^i, \theta_i) \geqq u(x, y^i, \theta_i)$ for all i with strict inequality for some i. We may now state the following well-known lemma without proof.

Lemma. *A feasible allocation* $\boldsymbol{z} = (x, y^1, \cdots, y^n)$ *is Pareto efficient if and only if it belongs to* $\boldsymbol{z}$ *and satisfies*

$$x\left[\sum_i \pi(x, y^i, \theta_i) - \gamma(x)\right] = 0, \quad x \geqq 0, \quad \sum_i \pi(x, y^i, \theta_i) - \gamma(x) \leqq 0.$$

4. Equity and Fairness

Let us consider the equity concept for the economy described in the previous section. As I mentioned in the introduction, a straightforward extension of the concept of equitable net trade for the private goods to our economy trivially leads to the recommendation that each individual should share the same amount of costs, even though it is easily understood that the amount of benefits each individual enjoys is different from the others according to the differences in their characteristics. So, first of all, we may need to measure the amount of benefits relative to the initial distribution of goods. One such suitable measure is the concept of consumer's surplus. It is well known that there are two concepts of consumer's surplus with respect to quantity variations proposed and investigated by Hicks in his celebrated articles ([9], [10] and [11]). What is considered here, however, is only the "equivalent surplus" referred to by Hause [7], which corresponds to Hicks's "quantity-equivalent variation."

Formally, the equivalent surplus relative to the initial holdings of goods for individual i, given his consumption bundle (x, y^i), is defined as the amount of private good $s^i = s^i(x, y^i)$ such that

(2) $\quad u(0, w^i + s^i, \theta_i) = u(x, y^i, \theta_i)$.

Of course, $s^i(0, w^i) = 0$ for all i. It is obvious that this consumer's surplus s^i corresponds in a one-to-one manner to the utility level, and that the higher the utility level, the larger is s^i (see Figure 1). Moreover, by virtue of Assumptions 2 and 3, the s^i's corresponding to feasible allocations are finite and continuously differentiable with respect to (x, y^i). Hereafter, the equivalent surplus will be referred to simply as consumer's surplus.

Note that the consumer's surplus of individual i can be meaningfully compared with anyone else's without any reference to "measurable" utility. Given an allocation $\boldsymbol{z}$, if $s^j > s^i$ for some i and j, then it is said that i has a legitimate complaint against j. If no individual has a legitimate complaint against anyone else, the allocation may be called equitable. More formally,

Definition 1. An allocation $\boldsymbol{z} = (x, y^1, \cdots, y^n)$ is said to be *equitable in terms of consumer's surplus* (ECS) if the corresponding consumer's surpluses $(s^1(x, y^1), s^2(x, y^2), \cdots, s^n(x, y^n))$ are such that

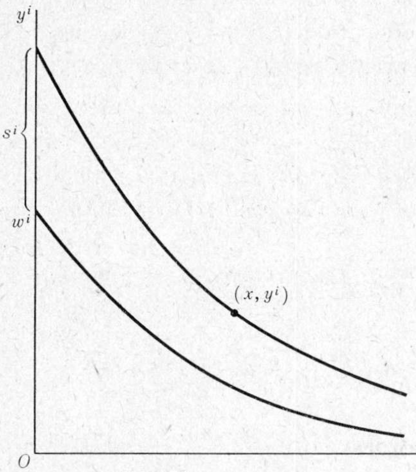

Figure 1. Equivalent Surplus $s^i = s^i(x, y^i)$

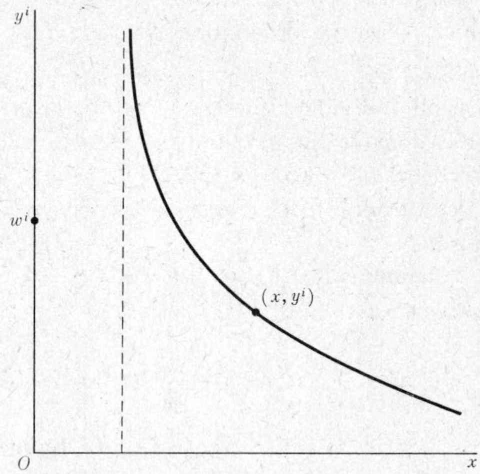

Figure 2.

$$s^i(x, y^i) = s^j(x, y^j) \quad \text{for all } i \text{ and } j.$$

We also make the following definition.

Definition 2. An allocation $\boldsymbol{z}$ is said to be *fair in terms of consumer's surplus* (FCS) if it is Pareto efficient and ECS.

Of course, this last definition would be vacuous if the FCS allocations fail to exist. Fortunately, however, we can show the following theorem.

Theorem 1. *Under Assumptions* 1, 2 *and* 3, *there exists a* FCS *allocation, which is uuique and individually rational.*

In the rest of this section, we will reconsider two assumptions. The first we would like to discuss is Assumption 3. If it were not for this assumption, there would be the possibility that the consumer's surplus s^i becomes infinitely large. Figure 2 illustrates such a situation where the dotted line depicts the asymptotic line for the indifference curve through (x, y^i). If the consumer's surplus s^i and s^j are infinite for some i and j, the comparison between them makes no sense. In order to avoid such a situation, it would be necessary to redefine the consumer's surplus.

Secondly, we examine the assumption that the individuals differ only in their observable characteristics: i. e., $u^i(\cdot, \cdot) = u(\cdot, \cdot)$ for all i. If we relax this assumption and admit the differences in preferences, the arguments would be more complicated.

First of all, we may need to restate the definition of ECS allocations. Given a consumption bundle (x, y^j), the consumer's surplus for individual j assessed by i, $s_i{}^j$, is defined as the amount of private good such that

$$u^i(0, w^j + s_i{}^j, \theta_j) = u^i(x, y^j, \theta_j).$$

Given an allocation $\boldsymbol{z}$, if $s_i{}^j > s_i{}^i$ for some i and j, then it is said that i has a legitimate complaint against j. If no individual has a legitimate complaint against anyone else, i. e., if $s_i{}^i \geqq s_i{}^j$ for all i and j, the allocation is said to be ECS. Moreover, an allocation which is Pareto efficient and ECS is said to be FCS.

The most troublesome problem in this general case is that the existence of FCS allocations is not necessarily warranted. In fact, it is quite easy to give an example in which there is no FCS allocation. It would, therefore, be necessary to rank all individually rational feasible allocations from the viewpoint of distributive fairness according to some private estimation of them, which might be formulated by the n^2 numbers s_i^j's. Such an attempt has been made by Otsuki [14] for a productive economy without public goods.

5. A Planning Procedure

In this section, we will consider what kind of mechanism and planning procedure implements a FCS allocation, which was shown to exist in the last section. The existing literature on incentive compatibility tells us that the fact FCS allocation being individually rational and Pareto efficient makes the problem quite tractable. For example, let us consider the MDP procedure proposed by Drèze and de la Vallée Poussin [5] and Malinvaud ([12] and [13]). Champsaur [2] showed that this planning procedure has an important property called neutrality: i. e., any individually rational and Pareto efficient allocation, and hence the FCS allocation, would be asymptotically attained by a suitable choice of parameters incorporated in the procedure which prescribe the share of social surplus to each individual. However, the planning bureau must decide such parameters *a priori*. Thus it is presumed that the planning bureau is accumulating exhaustive information as to individual preferences in deciding such parameters. We will, therefore, propose another continuous time planning procedure which is analogous to the MDP procedure.

The FCS allocation has two independent aspects, efficiency and equity. The planning bureau, therefore, has to ask the individuals to report information on two variables. In more detail, at any time $t(\geqq 0)$ and given an allocation $\boldsymbol{z}(t)=(x(t), y^1(t), \cdots, y^n(t))$, individuals are asked to report $(\pi^i(t), \sigma^i(t))$. Here $\pi^i(t)$ is the marginal rate of substitution of individual i evaluated at $(x(t), y^i(t))$, i. e., $\pi(x(t), y^i(t), \theta_i)$, and $\sigma^i(t)=\sigma^i(x(t), y^i(t))$ describes the increment of private good which is necessary to increase the consumer's surplus of individual i by the marginal amount while the amount of public good being kept unchanged. More formally,

$$(3)\quad \sigma^i(x(t), y^i(t)) := \left[\frac{\partial}{\partial y} s^i(x(t), y^i(t))\right]^{-1},$$

where $s^i(t)=s^i(x(t), y^i(t))$ is the consumer's surplus defined in (2). Due to Assumption 2, $s^i(t)$ is continuous in $(x(t), y^i(t))$, and so is $\sigma^i(t)$. This assumption also guarantees that $\sigma^i(t)$ is always positive.

The planning bureau then revises the allocation $\boldsymbol{z}(t)$ according to

$$(4)\quad \dot{x}(t)=\begin{cases}\sum_i \pi^i(t)-\gamma(t) & \text{if } x(t)>0,\\ \max\left[0, \sum_i \pi^i(t)-\gamma(t)\right] & \text{if } x(t)=0,\end{cases}$$

$$(5)\quad \dot{y}^i(t)=-\pi^i(t)\dot{x}(t)+\frac{\sigma^i(t)}{\sum_j \sigma^j(t)}\left[\sum_j \pi^j(t)-\gamma(t)\right]\dot{x}(t),\quad i\in N,$$

where the upper dot denotes the right hand side derivative with respect to time. $\gamma(t)$ is used as an abbreviation for $\gamma(x(t))$.

These adjustment rules constitute a system of ordinary differential equations defined on R_+^{n+1}. In fact, (4) is designed so that the amount of public good is always nonnegative.

On the other hand, the amount of private good allotted to each individual is nonnegative too. For if $y^i(t)=0$ for some i and t, then we have $\pi^i(t)=0$ by Assumption 2 and hence $\dot{y}^i(t)\geqq 0$ by (4) and (5).

Even though the right hand side of (4) is not continuous when the quantity of public good is zero, it can be shown that there is at least one continuous solution path in R_+^{n+1} starting from $\boldsymbol{z}_0\in R_+^{n+1}$.[1)] In particular, if $\boldsymbol{z}_0\in\bar{Z}$, the solution path starting from $\boldsymbol{z}_0$ is contained in $\bar{Z}$, since (4) and (5) satisfy $\gamma(t)\dot{x}(t)+\sum_i\dot{y}^i(t)=0$. The next theorem states some further important properties of the solution path.

Theorem 2. *Let* $\boldsymbol{z}_0=(x_0, y_0^1, \cdots, y_0^n)\in\bar{Z}$ *be an allocation such that* $s^i(x_0, y_0^i)=s^j(x_0, y_0^j)$ *for all* i *and* j. *Then, under Assumptions* 1, 2 *and* 3, *the solution path* $\{\boldsymbol{z}(t)\mid t\geqq 0\}$ *starting from* $\boldsymbol{z}_0$ *has the following properties. That is, along the path,*

i) *the utility level (and hence the consumer's surplus) of each individual continues to increase until a Pareto efficient allocation is attained,*

and in particular,

ii) *at each time, the amount of consumer's surplus accruing to each individual is the same across them.*

Moreover,

iii) *the allocation at the stationary point of the procedure is always a* FCS *allocation.*

As a typical example of the feasible allocation $\boldsymbol{z}_0$ which satisfies the condition in the theorem, we can always give the initial distribution of goods, $(0, w^1, \cdots, w^n)$.

Finally, we show the following theorem which states that the procedure is stable.

Theorem 3. *Let* $\boldsymbol{z}_0=(x_0, y_0^1, \cdots, y_0^n)\in\bar{Z}$ *be an allocation such that* $s^i(x_0, y_0^i)=s^j(x_0, y_0^j)$ *for all* i *and* j. *Then, under Assumptions* 1, 2 *and* 3, *the solution path starting from* $\boldsymbol{z}_0$ *generated by* (4) *and* (5) *always converges to the unique* FCS *allocation.*

6. Proofs

This section is devoted to the proofs of Theorems.

The proof of Lemma may be omitted.

Proof of Theorem 1. In order to show that the FCS allocations exist, let us consider the following problem:

1) The existence of solution paths to (4) and (5) is already shown by Henry [8] (see also Champsaur, Drèze and Henry [3]). But his proof requires some additional assumptions on the individual preferences and on the production technology. So, we will give another simple proof here. We will first extend the system of ordinary differential equations (4) and (5) defined on R_+^{n+1} to that defined on $R\times R_+^n$ as Henry did. For any $\boldsymbol{z}=(x, y^1, \cdots, y^n)\in R\times R_+^n$, define

$$\dot{x}=\sum_i\pi(g(x), y^i, \theta_i)-\gamma(g(x)),$$

$$\dot{y}^i=-\pi(g(x), y^i, \theta_i)\dot{x}+\frac{\sigma^i(g(x), y^i)}{\sum_j\sigma^j(g(x), y^j)}\left[\sum_j\pi(g(x), y^j, \theta_j)-\gamma(g(x))\right]\dot{x}$$

for all $i\in N$, where $g(x)=\max[0, x]$. Since the right hand sides of these equations are all continuous, the fundamental existence theorem assures the existence of at least one continuous solution path in $R\times R_+^n$. Now consider a solution path $\{\bar{\boldsymbol{z}}(t)\mid\bar{\boldsymbol{z}}(t)=(\bar{x}(t), \bar{y}^1(t), \cdots, \bar{y}^n(t)), t\geqq 0\}$ starting from $\boldsymbol{z}_0\in R_+^{n+1}$ and let $T=\inf\{t\mid\bar{x}(t)<0\}$. If we define a new path $\{\boldsymbol{z}(t)\mid t\geqq 0\}$ as

$$\boldsymbol{z}(t)=\begin{cases}\bar{\boldsymbol{z}}(t) & (t\leqq T),\\ \bar{\boldsymbol{z}}(T) & (t>T),\end{cases}$$

it is easy to verify that this new path is a solution to (4) and (5). Therefore, we can conclude that (4) and (5) have at least one continuous solution path.

(6) $\max_{z} s$

$$\text{subject to} \begin{cases} u(0, w^i+s, \theta_i) = u(x, y^i, \theta_i) \text{ for all } i \in N, \\ \boldsymbol{z} = (x, y^1, \cdots, y^n) \in Z, \\ s \geqq 0. \end{cases}$$

This optimization problem maximizes consumers' surpluses within the constraints that the corresponding allocation belongs to the feasible set and that all individuals enjoy the same nonnegative amount of consumer's surplus with each other. It is clear that any FCS allocation is a solution to this optimization problem.

The existence of a solution to this problem can be shown as follows. First, for simplicity, let $v^i(s) := u(0, w^i+s, \theta_i)$ and $\boldsymbol{z}_i = (x, y^i)$ given $\boldsymbol{z} = (x, y^1, \cdots, y^n)$. Obviously, $v^i: R \to R$ is continuous and strictly increasing by Assumption 2. Secondly, let $u(Z)$ be the image of Z under the mapping $u(\cdot) = (u(\cdot, \theta_1), \cdots, u(\cdot, \theta_n))$, and $v(R_+)$ be that of R_+ under the mapping $v(\cdot) = (v^1(\cdot), \cdots, v^n(\cdot))$. Since Z is a compact subset of R_+^{n+1} and u is continuous, the image $u(Z)$ is also a compact subset of R^n. By the same reason, $v(R_+)$ is a closed subset of R^n. Hence the intersection $u(Z) \cap v(R_+)$ is nonempty and compact, the nonemptiness being trivial by referring to the initial distribution of goods $(0, w^1, \cdots, w^n)$.

Now, without loss of generality, consider the projection function p_1 of R^n into R which assigns the first element to any point in R^n. Since p_1 is continuous, it attains a maximum on $u(Z) \cap v(R_+)$: i. e., there exist some $\boldsymbol{z}^* \in Z$ and $s^* \in R_+$ satisfying $u(\boldsymbol{z}_i^*, \theta_i) = v^i(s^*)$ for all i such that

$$v^1(s^*) \geqq v^1(s)$$

for any $(\boldsymbol{z}, s) \in Z \times R_+$ satisfying $u(\boldsymbol{z}_i, \theta_i) = v^i(s)$ for all i. By taking consideration of the fact that v^1 is a strictly increasing function, it would be understood that $\boldsymbol{z}^* \in Z$ is a solution to (6).

Note that

(7) $y^{i*} > 0$ for all $i \in N$.

In fact, if $y^{i*} = 0$ for some i, then it follows from Assumption 2 that $u(x^*, y^{i*}, \theta_i) = u(0, 0, \theta_i) < u(0, w^i+s^*, \theta_i)$, which contradicts the fact that $u(x^*, y^{i*}, \theta_i) = u(0, w^i+s^*, \theta_i)$.

In the sequel, we will show that the allocation $\boldsymbol{z}^* = (x^*, y^{1*}, \cdots, y^{n*})$, which is a solution to (6), is the one which satisfies all the properties in the theorem. First, we show that $\boldsymbol{z}^*$ is a FCS allocation. Obviously, $\boldsymbol{z}^*$ is ECS, since all individuals enjoy the same amount of consumer's surplus s^* at $\boldsymbol{z}^*$. In order to show that $\boldsymbol{z}^*$ is Pareto efficient, suppose that there is some other feasible allocation $\boldsymbol{z} = (x, y^1, \cdots, y^n)$ which is more efficient than $\boldsymbol{z}^*$: i. e., $\boldsymbol{z}$ satisfies $u(x, y^i, \theta_i) \geqq u(x^*, y^{i*}, \theta_i)$ for all i and $u(x, y^j, \theta_j) > u(x^*, y^{j*}, \theta_j)$ for some j. Here note that $y^j > 0$. In fact, if $y^j = 0$, then we have from Assumption 2 that $u(x, y^j, \theta_j) = u(0, 0, \theta_j) \leqq u(x^*, y^{j*}, \theta_j)$, which is a contradiction. Therefore, since the utilities are assumed to be strictly increasing in the amount of the private good, it holds for $\varepsilon > 0$ and sufficiently small that $u(x, y^j - \varepsilon, \theta_j) > u(x^*, y^{j*}, \theta_j)$. By the same reason, we also have $u(x, y^i + \varepsilon/(n-1), \theta_i) > u(x^*, y^{i*}, \theta_i)$ for all $i \neq j$. Thus, if we define $\{s^h\}$ as $u(0, w^j + s^j, \theta_j) = u(x, y^j - \varepsilon, \theta_j)$ and $u(0, w^i + s^i, \theta_i) = u(x, y^i + \varepsilon/(n-1), \theta_i)$ for all $i \neq j$, then we have $s^h > s^*$ for all $h \in N$. Now let $\bar{s} = \min_h \{s^h\}$ and $\{y^{h\prime}\}$ be the amounts of the private good such that $u(x, y^{h\prime}, \theta_h) = u(0, w^h + \bar{s}, \theta_h)$ for all $h \in N$. Obviously $\bar{s} > s$. Moreover, it is easily verified that $0 < y^{j\prime} \leqq y^j - \varepsilon$ and $0 < y^{i\prime} \leqq y^i + \varepsilon/(n-1)$ for all $i \neq j$, so that this new allocation $\boldsymbol{z}' = (x, y^{1\prime}, \cdots,$

$y^{n\prime}$) satisfies (1). We can, therefore, attain at $\boldsymbol{z}'$ the consumer's surplus $\bar{s}$ common to all individuals. This $\bar{s}$ is greater than s^* obtained at $\boldsymbol{z}^*$, which contradicts the supposition that $\boldsymbol{z}^*$ is a solution to (6).

Secondly, the allocation $\boldsymbol{z}^*$ is individually rational since $s^* \geqq 0$.

Finally, the uniqueness of the FCS allocation would be shown by following the familiar procedure. Since we have shown that a feasible allocation is FCS if and only if it is a solution to (6), it is sufficient to prove the uniqueness of the solution. Let $\boldsymbol{z}_1 = (x_1, y_1^1, \cdots, y_1^n)$ and $\boldsymbol{z}_2 = (x_2, y_2^1, \cdots, y_2^n)$ be two different feasible allocations which are the solutions to (6) corresponding to the same maximal consumer's surplus s^*. And define a new allocation $\boldsymbol{z}(\alpha) = (x(\alpha), y^1(\alpha), \cdots, y^n(\alpha))$ for $\alpha \in (0, 1)$ as

$$\boldsymbol{z}(\alpha) = \alpha \boldsymbol{z}_1 + (1-\alpha)\boldsymbol{z}_2.$$

Trivially, $z(\alpha) \in Z$ by Assumption 1. Moreover, $y^i(\alpha) > 0$ for all i by (7). Since $\boldsymbol{z}(\alpha)$ is a convex combination of $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ with $\boldsymbol{z}_1 \neq \boldsymbol{z}_2$, and since the utilities are assumed to be strictly quasi-concave in the interior of R_+^2 and strictly increasing in the amount of private good (Assumption 2), $\boldsymbol{z}(\alpha)$ gives higher utility levels or equivalently larger consumer's surplus to each individual than $\boldsymbol{z}_1$ or $\boldsymbol{z}_2$ does. This contradicts the supposition that $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are solutions to (6).

Proof of Theorem 2. Following the adjustment rules (4) and (5), the change in utility levels may be expressed as

$$\dot{u}(x, y^i, \theta_i) = u_y(x, y^i, \theta_i)\frac{\sigma^i}{\sum_j \sigma^j}\dot{x}^2 \quad \text{for all} \quad i \in N.$$

Here we already know that $u_y(x, y^i, \theta_i)$ and σ^i's are always positive. Moreover, since $\boldsymbol{z}_0 \in \bar{Z}$ and hence $\boldsymbol{z}(t) \in \bar{Z}$ for all $t \geqq 0$, Lemma implies that $\dot{x} = 0$ if and only if the corresponding allocation $\boldsymbol{z}$ is Pareto efficient. Thus i) holds trivially.

In order to prove ii), it is sufficient to show that $\dot{s}^i(t) = \dot{s}^j(t)$ for all i, j and t, since $s^i(0) = s^j(0)$ for all i and j. But by virtue of (2) and (3), we have

$$\dot{s}^i(t) = \frac{u_y(x(t), y^i(t), \theta_i)}{u_y(0, w^i + s^i(t), \theta_i)} \cdot \frac{\sigma^i(t)}{\sum_j \sigma^j(t)} \cdot [\dot{x}(t)]^2 = \frac{1}{\sum_j \sigma^j(t)} \cdot [\dot{x}(t)]^2,$$

which guarantees that $s^i(t) = s^j(t)$ for all i, j and t.

i) and ii) shown above imply that the procedure terminates at some time t if and only if the corresponding allocation $\boldsymbol{z}(t)$ is Pareto efficient, while the consumers' surpluses are the same across individuals. Thus the stationary point of the procedure must be Pareto efficient and ECS, and hence must be a FCS allocation.

Proof of Theorem 3. Due to iii) of the previous theorem, we know that the stationary point of the procedure is always a FCS allocation which is unique and individually rational (Theorem 1). Thus, by referring to the Lyapunov second method, it is sufficient to construct a Lyapunov function $V: R_+ \rightarrow R: t \rightarrow V(t)$ with the following properties:

i) $V(\cdot)$ is continuous and strictly monotone increasing in t,

ii) $\dot{V}(t) = 0$ if and only if the corresponding allocation $\boldsymbol{z}(t)$ is a stationary point of the procedure.

Now consider such a function

$$V(t) := \sum_i u(x(t), y^i(t), \theta_i).$$

Theorem 2 assures that V satisfies i) and ii) listed above, which completes the proof.

(Faculty of Economics, Tohoku University)

References

[1] Berge, C., *Topological Spaces*, Tr. by E. M. Patterson (Oliver and Boyd, London), 1963.

[2] Champsaur, P., "Neutrality of Planning Procedures in an Economy with Public Goods," *Review of Economic Studies* 43, 1976, 293–300.

[3] Champsaur, P., J. H. Drèze and C. Henry, "Stability Theorems with Economic Applications," *Econometrica* 45, 1977, 273–294.

[4] Daniel, T. E., "Pitfalls in the Theory of Fairness——Comment," *Journal of Economic Theory* 19, 1978, 561–564.

[5] Drèze, J. H. and D. de la Vallée Poussin, "A Tâtonnement Process for Public Coods," *Review of Economic Studies* 38, 1971, 133–150.

[6] Foley, D. K., "Resource Allocation and the Public Sector," *Yale Economic Essays* 7, 1967, 45–98.

[7] Hause, J. C., "The Theory of Welfare Cost Measurement," *Journal of Political Economy* 83, 1975, 1145–1182.

[8] Henry, C., "Differential Equations with Discontinuous Right-hand Side for Planning Procedures," *Journal of Economic Theory* 4, 1972, 545–557.

[9] Hicks, J. R.,"Consumer's Surplus and Index Numbers," *Review of Economic Studies* 9, 1942, 126–137.

[10] Hicks, J. R.. "The Four Consumer's Surpluses," *Review of Economic Studies* 11, 1943, 31–41.

[11] Hicks, J. R., "The Generalized Theory of Consumer's Surplus," *Review of Economic Studies* 13, 1945–1946, 68–74.

[12] Malinvaud, E., "A Planning Approach to the Public Good Problem," *Swedish Journal of Economics* 73, 1971, 96–112.

[13] Malinvaud, E., "Prices for Individual Consumption, Quantity Indicators for Collective Consumption," *Review of Economic Studies* 39, 1972, 385–405.

[14] Otsuki, M., "On Distribution According to Labour——A Concept of Fairness in Production Economies," *Review of Economic Studies* 47, 1980, 945–958.

[15] Pazner, E. A. and D. Schmeidler, "A Difficulty in the Concept of Fairness," *Review of Economic Studies* 41, 1974, 441–443.

[16] Pazner, E. A. and D. Schmeidler, "Egalitarian Equivalent Allocation——A New Concept of Economic Equity," *Quarterly Journal of Economics* 92, 1978, 671–687.

[17] Schmeidler, D. and K. Vind, "Fair Net Trades," *Econometrica* 40, 1972, 637–642.

[18] Thomson, W., "An Informationally Efficient Equity Criterion," *Journal of Public Economics* 18, 1982, 243–263.

[19] Varian, H. R., "Envy, Equity and Efficiency," *Journal of Economic Theory* 9, 1974, 63–91.

[20] Varian, H. R., "Distributive Justice, Welfare Economics, and the Theory of Fairness," *Philosophy and Public Affairs* 4, 1974–1975, 223–247.