The Burden of Debt and Intergenerational Distribution*

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1. Introduction

Who bears the burden of debt? It is an old and recurrent question in ecoomics. It seems that answer to the question depends upon the definition of the burden of debt. In this paper, we define the burden of debt to be a decrease in utility level of future generations (in the long run), which does not exist if the government debt were not issued.

Modigliani [7] argued that a permanent increase in governrent debt will displace the same amount of capital from private portfolio in the long run (see also Phelps and Shell [10]). However, his argument is confined only within stationary states and stability question is completely ignored. Hence, it remains unsettled whether the burden of debt may exist in that the economy may be unstable and debt issue may decrease private capital forever.

Diamond [5], using Samuelsonian generation overlapping framework, showed that not only an increase in government debt will decrease long run utility level of consumers but also these equilibria are stable. However, he considered an economy where the amount of tax is controlled so that the per capita government debt is held constant. Such a formulation does not conform to recent literature in another important branch in the theory of public debt; the question of crowding out (e. g., see Blinder and Solow [3]). It is also against our intuition. That is, the tax instruments are subject to congressional approval and are seldom altered, whereas government issues debt to finance budgetary deficit whenever necessary.

We do not believe that government always holds tax rates constant even when the economy moves to a crisis. We believe, however, that it is important to know what will happen to an economy when government fails to adjust the tax instruments in the face of the accumulation of large debt.

In a model without capital, i. e., in consumption-loan models, Gale [6] showed that the long-run competitive equilibrium without government debt (or Social Contrivance of Money) is likely to be stable but Pareto inefficient while the long-run equilibrium with debt is likely to be efficient but unstable if the economy is in what he called "Samuelsonian case" (see also Cass, Okuno and Zilcha [4]. In this regard, stability question of equlibria a là Diamond must be investigated more carefully.

In this paper, we will examine the stability property of long-run equilibria with capital when tax rate is predetermined and government debt issue is endogenously

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determined by budgetary deficit.

Recently Barro [1] showed that, when inheritance is allowed and each consumer cares the welfare of his descendants, issuance of government's debt will have no effect whatsoever to the real aspects of the economy. We do not follow his line of argument because we do not believe that consumers care the well-being of their descendants in the distant future, which is the key to his argument. We feel that the reality lies somewhere between his specification and ours where no consumer is concerned with the welfare of his descendants at all.

In section 2, the basic model will be formulated. In section 3, it will be shown that there are, in general, two types of long-run equilibria; stable inefficient equilibria and efficient saddle point equilibria. As a corollary, it will be shown that an exceedingly large debt issue will cause a burden in the future generations in either making government bankrupt or decumulating private capital forever. In section 4, the effects of a social security program will be studied. In section 5, an adaptive expectation formation will be considered instead of rational expectation formation. In both sections 4 and 5, the basic conclusions in section 3 will be shown to hold as well. Section 6 concludes the paper.

2. Model

The model is basically that of Diamond's. Namely, the economy under consideration has discrete but infinite future with an unchanging technology which has usual neoclassical production function, $F(K, L)$. Following Diamond, we assume that capital does not depreciate and we denote the net output by $F(K, L)$. Consumers in this economy live for two periods, working in the first period and being retired in the second. Each consumer has the identical utility function $U(e^1, e^2)$ which is a function of his consumption in the first period, e^1, and in the second, e^2. The number of young consumers or labor force in period t, L_t, is assumed to grow exponentially, i. e., $L_t = L_0(1+n)^t$.

A consumer born in period t makes his optimal decision given his disposable income $y_t{}^d$, and the rate of interest between t and $t+1$, r_{t+1}, in the following manner:

maximize $U(e_t{}^1, e_{t+1}{}^2)$

subject to $e_t{}^1 = y_t{}^d - s_t$,

$e_{t+1}{}^2 = (1+r_{t+1})s_t$,

where s_t is the amount of saving measured in terms of good. Then his consumption will be allocated to satisfy

$$\frac{\partial U}{\partial e_t{}^1} = (1+r_{t+1})\frac{\partial U}{\partial e_{t+1}{}^2},$$

Therefore, his saving s_t can be expressed as

(1) $s_t = s(y_t{}^d, r_{t+1})$.

We assume that s is continuously differentiable in both arguments. Assuming consumptions to be normal, $0 < \frac{\partial s}{\partial y^d} < 1$. However, the sign of $\frac{\partial s}{\partial r}$ depends upon the relative magnitude of income and substitution effects.

We now turn to the demand side of the capital market. In period t, capital is demanded for the use in period $t+1$. Therefore, its demand depends upon the expectation of prevailing wage level in period $t+1$, w_{t+1}. In this section, partly because of the simplicity of exposition and partly following Diamond, we assume that the expectation of w_{t+1} is formed rationally, or by perfect foresight. Then assuming perfect labor market and perfect competition in capital market,

$$r_{t+1}=\partial F(K_{t+1}, L_{t+1})/\partial K_{t+1}.$$

Since the production function is of constant returns to scale, $F(K, L)$ can be written as $Lf(k)$ where k is the capital labor ratio, and

$$(2) \quad r_{t+1}=f'(k_{t+1}).$$

Constant returns production function, at the same time, enables us to denote the relationship between marginal products of both capital and labor as

$$(3) \quad w=\phi(r) \quad \text{where } w=f(k)-kf'(k) \text{ and } r=f'(k).$$

Moreover,

$$\frac{dw}{dr}=\phi'(r)=-k<0 \text{ and}$$

$$\frac{d^2w}{dr^2}=\phi''(r)=\frac{-1}{f''(k)}>0.$$

There is another participant in the demand side of the capital market; government. Basically, government's role is to achieve intergenerational transfer with the help of tax and debt instruments. For the simplicity, we assume that government debt is of one period maturity. In particular, we assume that government issues debt in period 1 to subsidize the old generation in the period with its proceeds. Form the period 2 on, government finances the repayment of outstanding debt by either issuing new debt, imposing tax on young generation or both. Although this story seems too restrictive, it is similar to installing a pay-as-you-go social security system. The question of a social security system will be discussed more fully in subsequent section.

Let the government debt issue in period $t-1$ be G_{t-1}. The obligation of government in the beginning of period t, $(1+r_t)G_{t-1}$ is paid by predetermined tax T_t and new issue of public debt G_t. Denoting the end of period debt per capita by g and tax per capita by $\hat{t}$,

$$(4) \quad (1+r_t)g_{t-1}=(1+n)(g_t+\hat{t}_t).$$

In this section, we consider the initial debt issue (subsidy to the old generation in period 1), g_1, and proportional tax rate, τ, to be the only policy variables which government can control. Assuming taxable income consists only of wage income (when young), (4) becomes[1)]

$$(4)' \quad (1+r_t)g_{t-1}=(1+n)(g_t+\tau w_t).$$

In the subsequent analyses, we allow g_t to be negative as well. Negative g_t means government's loan to consumers at the market rate of interest[2)].

1) We can introduce tax on interest income without altering our model in any significant way. Specifically, let $\hat{\tau}$ be the tax rate on interest income, $\hat{r}$ be the before tax rate of interest, and r be the after tax rate of interest, i. e., $r=(1-\hat{\tau})\hat{r}$. Then, all the equations follow except (4)-(4)'' are altered. E. g., (4)'' must be altered so that

$$(4)''' \quad g_t-g_{t-1}=\frac{r_t-n}{1+n}g_{t-1}-(\tau w_t+\hat{\tau}r_tk_t).$$

Our conclusions remain unaltered qualitatively.

2) The model can be modified easily, without changing the qualitative result, so that when government

Finally, capital market is in equilibrium when private saving equals the sum of private capital formation and governmental borrowing. In per capita terms, equilibrium condition becomes

(5) $s_t = s((1-\tau)w_t, r_{t+1}) = (1+n)k_{t+1} + g_t.$

In view of (2), (5) can be rewritten as

(6) $r_{t+1} = f'\left(\dfrac{s_t - g_t}{1+n}\right).$

3. Dynamic Behavior of the Model

We can illustrate dynamic behavior of the economy described by (1)-(6) in a phase diagram[3]. For this purpose we rewrite (4)′ as

(4)″ $g_t - g_{t-1} = \dfrac{r_t - n}{1+n} g_{t-1} - \tau w_t.$

Hence if either (a) $g_{t-1} > 0$ and $r_t \leqq n$ or (b) $g_{t-1} < 0$ and $r_t \geqq n$, then $g_t < g_{t-1}$. To analyze the behavior of g_t in other cases, we first find the locus of (g, r) where $g_t = g_{t-1}$. Then this locus must satisfy

(7) $g = \dfrac{1+n}{r-n} \tau \phi(r) \equiv \gamma(r).$

This locus, illustrated in Figure 1, has downward slope when $g_t > 0$ and $r_t > n$ as

$$\left.\frac{\partial g}{\partial r}\right|_{g_t=\text{const.}} = g\left[\frac{\phi'}{\phi} - \frac{1}{r-n}\right].$$

When $g_t < 0$ and $r_t < n$, the sign depends upon the relative magnitude of $\dfrac{\phi'}{\phi}$ and $\dfrac{1}{r-n}$. However, since $\gamma(r)$ approaches to $-\infty$ as $r \to n$, and $\gamma(r)$ approaches to a negative number ($-\infty$ if Inada condition is assumed) as $r \to 0$, $\gamma(r)$ is likely to be bell-shaped on this domain. By differentiating (4)″ partially with respect to g_{t-1},

$$\frac{\partial(g_t - g_{t-1})}{\partial g_{t-1}} = \frac{r_t - n}{1+n}.$$

Hence, when $r_t > n$, $g_t > g_{t-1}$ if and only if (r_{t-1}, g_{t-1}) lies above the locus of $g_t = g_{t-1}$. While when $r_t < n$, $g_t < g_{t-1}$ if and only if (r_{t-1}, g_{t-1}) lies above the locus of $g_t = g_{t-1}$.

On the other hand the locus of $r_t = r_{t-1}$ is determined by analyzing

(8) $r = f'\left(\dfrac{s((1-\tau)w, r) - g}{1+n}\right).$

By totally differentiating (8), we obtain the slope of the locus $r_t = r_{t-1}$ as

$$\left.\frac{\partial g}{\partial r}\right|_{r_t = r_{t-1}} = \frac{(1+n) - f''(k)\left[\dfrac{\partial s}{\partial r} + \dfrac{\partial s}{\partial y^d}(1-\tau)\phi'(r)\right]}{f''(k)}.$$

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debt issue, g, will never become negative. For example, one can introduce the government's behavior such that τ becomes zero when $g < 0$.

Although we do not allow public capital in our model, g_t may be more adequately interpreted as the difference between the amount of public debt and the amount of public capital. For this interpretation to be consistent with our model, however, we must introduce public capital explicitly.

3) Since our dynamic model is characterized by difference equations, strictly speaking, our analysis with the aid of a phase diagram is not appropriate and should be considered as a first order approximation.

Since the sign of the numerator cannot be determined, the sign of the entire expression is not unambiguous. However, if there is no government intervention (i. e., when $g=\tau=0$), the stability of free enterprise economy is guaranteed only when

$$(1+n) > \frac{ds(w(k), r(k))}{dk} = f''\left[\frac{\partial s}{\partial y^d}\phi' + \frac{\partial s}{\partial r}\right]$$

in the neighbourhood of the equilibrium. Hence we may assume thet $\left.\frac{\partial g}{\partial r}\right|_{r_t=r_{t-1}} < 0$ when g is close to zero. In the follwing, we assume, for all r and τ, that

$$(1+n) > f''\left[\frac{\partial s}{\partial r} + \frac{\partial s}{\partial y^d}(1-\tau)\phi'\right].$$ [4]

Under this assumption,

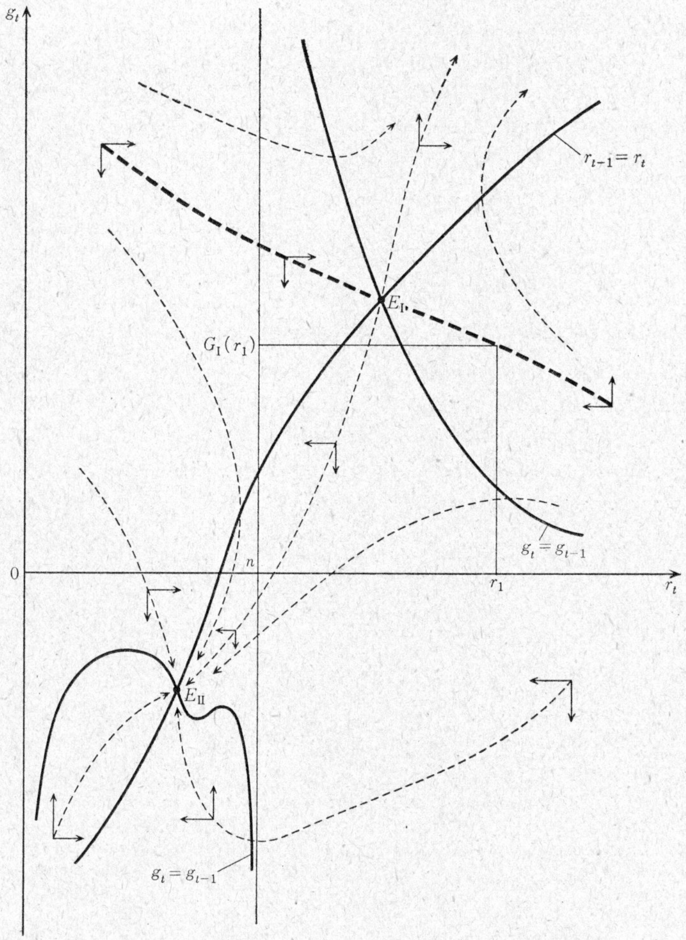

Figure 1

$$\frac{\partial(r_t - r_{t-1})}{\partial r_t} = \frac{f''\left[\frac{\partial s}{\partial r} + \frac{\partial s}{\partial y^d}(1-\tau)\phi'\right] - (1+n)}{(1+n) - f''\frac{\partial s}{\partial r}} < 0,$$

and

$$\frac{\partial(r_t - r_{t-1})}{\partial g_t} = \frac{f''}{(1+n) - f''\frac{\partial s}{\partial r}} > 0.$$

The resulting phase diagram is illustrated in Figure 1. There are, in general, two types of equilibrium which we labeled as E_{I} and E_{II}. Type I equilibrium has the rate

4) This is essentially what Diamond assumed for the stability of his model. The assumption is perhaps too strong to hold in any economy. However, for all the practical purposes, the inequality is needed to hold for a sufficiently large set of (r, τ). If, for example, both production function and utility function are of the Cobb-Douglous type, the inequality holds as long as $r < (1+n)/(1-\alpha)(1-\beta)$ where α is capital's share while β is the average propensity to consume.

of interest larger than the rate of population growth. Therefore, this equilibrium is Pareto efficient (see, e. g., Benveniste [2]). On the other hand, type II equilibrium has the property $r<n$ and hence inefficient. Moreover, in the long-run $g<0$ and government lends fund to public using tax revenue.

From stability viewpoint, type I equilibrium is always a saddle point and hence unstable except only one path. On the other hand, type II equilibrium is likely to be stable from the global viewpoint, though it may be locally unstable[5]. To be more precise, consider a time path, given the initial rate of interest, r_1, and the initial debt per capita, g_1,

$$\alpha(r_1, g_1{:}t) = (r_t, g_t)$$

which satisfies (4)′ and (5) for all $t'(1 \leqq t' \leqq t)$. Then for each r_1, there is an open unbounded set $G_{\mathrm{II}}(r_1) \subset R$ such that $(r_1, g_1{:}t)$ converges to a type II equilibrium if $g_1 \varepsilon G_{\mathrm{II}}(r_1)$ (and unless α forms a limit cycle around a type II equilibrium). In other words, type II equilibrium is quasi stable on $R_+ \times G_{\mathrm{II}}(R_+)$.

Define $G_{\mathrm{I}}(r_1)$ to be a set of g_1's such that $\alpha(r_1, g_1 : t)$ converges to a type I equilibrium. Then in this model (of sections 2 and 3), $G_{\mathrm{I}}(r_1) = \sup_{g_1} G_{\mathrm{II}}(r_1)$. Namely, regardless of the magnitude of r_1, there is a critical level of government debt issue in period 1, $G_{\mathrm{I}}(r_1)$. If debt issue in period 1, g_1, is larger than $G_{\mathrm{I}}(r_1)$ then the economy eventually approaches to a situation where there is too much debt outstanding and private capital formulation must be sacrificed for the repayment of the large government debt. On the other hand, if $g_1 < G_{\mathrm{I}}(r_1)$, then the economy will eventually converge to an equilibrium

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5) The local stability of the equilibria can be analyzed mathematically. From (4)″ and (6), we obtain, at the neighbourhood of an equilibrium,

$$\begin{bmatrix} d\hat{r}_{t+1} \\ d\hat{g}_{t+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} d\hat{r}_t \\ d\hat{g}_t \end{bmatrix}$$

where $d\hat{r}_t = r_t - r^*$, $d\hat{g}_t = g_t - g^*$ and (r^*, g^*) is an equilibrium. In the above expression

$$A = \frac{f'' \dfrac{\partial s}{\partial y^d}(1-\tau)\phi'}{1+n-f''\dfrac{\partial s}{\partial r}} > 0 \qquad (0<A<1),$$

$$B = \frac{f''}{1+n-f''\dfrac{\partial s}{\partial r}} > 0,$$

$$C = AE,$$

$$D = BE + \frac{1+r}{1+n},$$

where

$$E = \tau\phi\left[\frac{1}{r-n} - \frac{\phi'}{\phi}\right].$$

In order to find the property of the characteristic roots of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, define the characteristic equation

$$\phi(\lambda) = \lambda^2 - (A+D)\lambda + AD - BC.$$

When $r>n$ and $g<0$, it can be easily shown that $\phi(1)<0$ and $\phi(-1)>0$. Hence one of the roots is real and larger than unity while the other root has absolute value less than unity. Namely, any equilibrium with $r>n$ and $g<0$ is a saddle point.

When $r<n$, the equilibrium is locally stable if and only if

$$(1+A)\left(1+\frac{1+r}{1+n}\right) < BE < (1-A)\frac{n-r}{1+n}.$$

where too large capital labor ratio eats up consumption more than efficiently. Only when $g_1=G_{\text{I}}(r_1)$, the economy will converge to an efficient equilibrium but a slight disturbance will cause the economy either to go bankrupt or to approach to an inferior equilibrium.

When $\tau=0$, difference equations (4)′ and (6) can be written as

$$g_t-g_{t-1}=\frac{r_t-n}{1+n}g_{t-1},$$

$$r_t=f'\left(\frac{s(w_{t-1},r_t)-g_{t-1}}{1+n}\right).$$

Its phase diagram is illustrated in Figure 2 and the equilibrium E_{I} in this case $(\tau=0)$ coincides with the long-run equilibrium when government does not intervene (i. e., $g=\tau=0$).

If τ is a varibale, the location of equilibria depends upon τ. Substituting (7) into (8) and differentiating, we obtain

$$\frac{dr}{d\tau}=\frac{\frac{f''}{1+n}\left[\frac{1+n}{n-r}-\frac{s}{y^d}\right]\phi}{\alpha\left[1-A-\frac{BE}{n-r}\right]}$$

where

$$\alpha=1+n-f''\frac{\partial s}{\partial r},$$

$$A=f''\frac{\partial s}{\partial y^d}(1-\tau)\phi'/\alpha,$$

$$B=-f''/\alpha,$$

$$E=\tau\phi\left[\frac{1}{r-n}-\frac{\phi'}{\phi}\right].$$

Since $0<\frac{\partial s}{\partial y^d}<1$, $\left[\frac{1+n}{n-r}-\frac{\partial s}{\partial y^d}\right]$ has the same sign as that of $(n-r)$. Hence if $n<r$, $\frac{dr}{d\tau}>0$. On the other hand, if $n>r$, we cannot determine the sign of $\frac{dr}{d\tau}$ unambiguously. However, if the equilibrium is locally stable (see footnote 4),

$$\alpha\left[(1-A)\frac{n-r}{1+n}-BE\right]>0$$

and

$$\frac{dr}{d\tau}<0.$$

As the proportional tax rate, τ, increases, therefore, the equilibrium rate of interest increases when the rate, r^*, exceeds n. This is basically what Diamond obtained for internal debt case[6]. Under the assumption of normality of consumptions, one can easily show that the steady state utility will decrease as τ increases. On the other hand, the

6) Our model and Diamond's model are different first in the specification of tax. Namely, he assumes that the tax is collected so that the per capita debt is held constant while we assume that tax rate is held constant. The second difference lies in the definition of per capita debt holding, g_t. While our g_t is the end of the period debt G_t devided by the population of young consumers L_t, his per capita debt, g_t^D is defined to be the end of the period debt G_t devided by the population of young consumers in $t+1$, L_{t+1}. Hence $g_t=(1+n)g_t^D$.

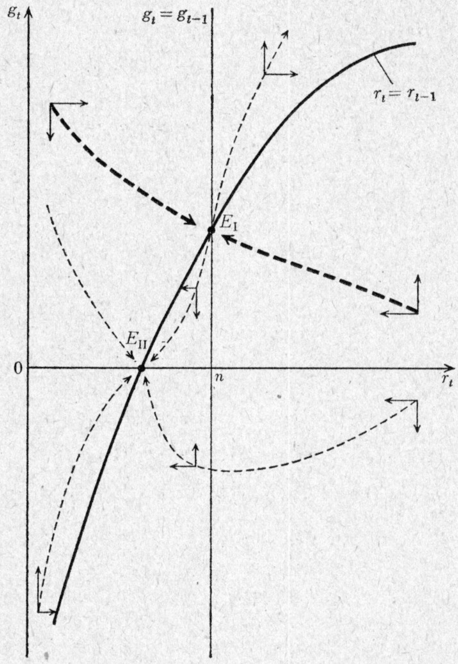

Figure 2

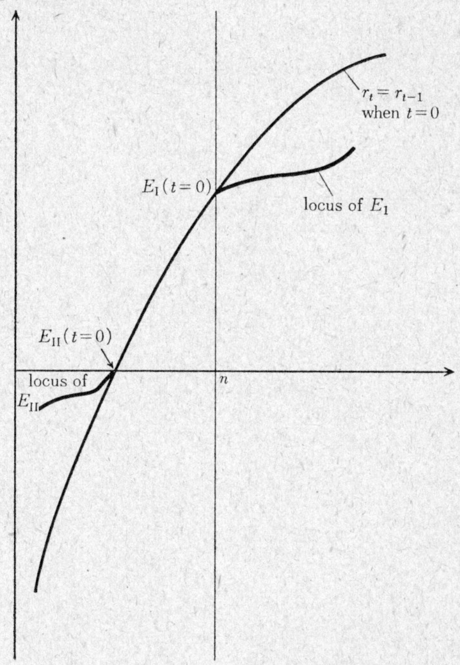

Figure 3

equilibrium of type II, (r^*, g^*), changes, as τ increases, so that r^* decreases. These analyses are illustrated in Figure 3.

4. Social Security Program

In the previous sections, we analyzed a model where government subsidizes the old generation in period 1 only. In this section, however, we consider a case where government subsidizes the old generation in each period by a predetermined amount per capita. Namely, let σ be the (per capita) amount of social security payment when old. Then in each period t, the total amount of social security is σL_{t-1}. On the other hand, the tax revenue is $\tau w_t L_t$. The difference is either financed by issuance of public debt $(G_t > 0)$ or loaned to the public $(G_t < 0)$ at the market rate of interest. Hence in per capita (young generation) terms,

(9) $\quad (1+r_t)g_{t-1} = (1+n)(g_t + \tau w_t) - \sigma,$

or

(9)′ $\quad g_t - g_{t-1} = \dfrac{r_t - n}{1+n} g_{t-1} - \tau w_t + \dfrac{\sigma}{1+n}.$

On the other hand, consumer's problem is now defined as

maximize $\quad U(e^1, e^2)$

subject to $\quad e_t^1 = (1-\tau)w_t - s_t$

$\quad e_t^2 = (1+r_{t+1})s_t + \sigma.$

Hence, saving is now a function not only of $(1-\tau)w_t$ and r_{t+1} but also of σ. Thus,

(10) $$r_{t+1}=f'\left(\frac{s((1-\tau)w_t,\sigma,r_{t+1})-g_t}{1+n}\right).$$

Note that $0<\dfrac{\partial s}{\partial y^d}=(1+r)\dfrac{\partial s}{\partial \sigma}<1.$

Define $\hat{\gamma}(r)$ as

(11) $$g=\frac{1+n}{r-n}\left[\tau\phi(\tau)-\frac{\sigma}{1+n}\right]\equiv-\hat{\gamma}(r)$$

and $\hat{r}$ as

(12) $$\tau\phi(\hat{r})=\frac{\sigma}{1+n}.$$

The shape of $\hat{\gamma}(r)$ depends upon whether $\hat{r}>n$ or not. When $\hat{r}<n$, $\hat{\gamma}(r)<0$ if $r>n$ or $r<\hat{r}$, and $\hat{\gamma}(r)>0$ if $\hat{r}<r<n$. Moreover $\hat{\gamma}(r)$ is increasing when $r\geqq\hat{r}$ (see Figures 6 and 7). On the other hand when $\hat{r}>n$, $\hat{\gamma}(r)<0$ if $r>\hat{r}$ or $r<n$, and $\hat{\gamma}(r)>0$ if $n<r<\hat{r}$. $\hat{\gamma}(r)$ is decreasing when $n<r\leqq\hat{r}$ (see Figures 4 and 5).

The locus of $r_t=r_{t-1}$

(13) $$r=f'\left(\frac{s((1-\tau)\phi(r),\sigma,r)-g}{1+n}\right)$$

and its slope $\left.\dfrac{dg}{dr}\right|_{r_t=r_{t-1}}$ are essentially the same as in the previous section except the inclusion of σ in the saving function. Thus we assume that $\left.\dfrac{dg}{dr}\right|_{r_t=r_{t-1}}$ is upward sloping as in the previous section.

The phase diagram and the stability property of equilibria depends upon whether $\hat{r}>n$ or not. It also depends upon the level of interest rate, $\bar{r}$, which clears the capital market when government debt is held zero. Namely,

$$\bar{r}=f'\left(\frac{s((1-\tau)\phi(\bar{r}),\sigma,\bar{r})}{1+n}\right).$$

Case 1. $\hat{r}>n$ and $\bar{r}<\hat{r}$ (Figure 4).
There are two types of equilibria. Type I equilibria ($r^*>n$, $g^*>0$) are always saddle, while type II equilibria ($r^*<n$, $g^*<0$) may be locally stable[7]. Moreover, defining $G_{\text{II}}(r_1)$ in the same manner as in the previous section, the process is quasi stable on $R_+\times G_{\text{II}}(R_+)$.

Case 2. $\hat{r}>n$ and $\bar{r}>\hat{r}$ (Figure 5).
Again, there are two types of equilibria. Type I equilibria ($r^*>n, g^*<0$) may be locally stable but likely to be a saddle point. Type II equilibria ($r^*<n$, $g^*<0$) have the same property as in the case 1. Defining $G_{\text{I}}(r_1)$ as the set of g_1's such that the economy will converge to a type I equilibrium, $G_{\text{I}}(r_1)$ may be multiple valued. However, $G_{\text{I}}(r_1)$ is always compact valued. Moreover, if g_1 does not belong to $G_{\text{II}}(r_1)$, the economy must either go bankrupt or approach to an inferior equilibrium.

Case 3. $\hat{r}<n$ and $\bar{r}<\hat{r}$ (Figure 6).
Tyepe II equilibria ($\hat{r}<r^*<n$ and $g^*>0$) have the same property as those of type II equilibria in case 2. Again the economy is quasi-stable on $R_+\times G_{\text{II}}(R_+)$.

Case 4. $\hat{r}<n$ and $\bar{r}>r$.
There may not exist an equilibrium at all. If there are (see Figure 7), then type I

7) The precise condition for local stability will be found in the next footnote.

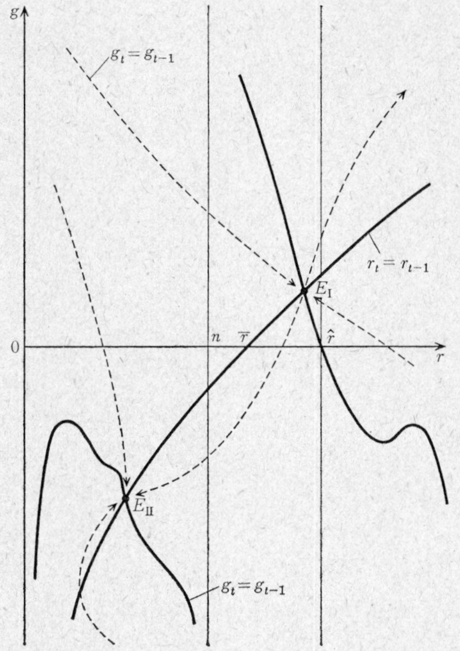

Figure 4

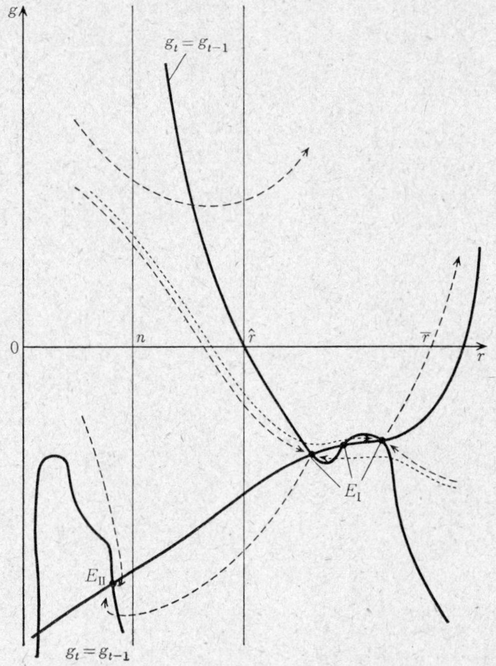

Figure 5

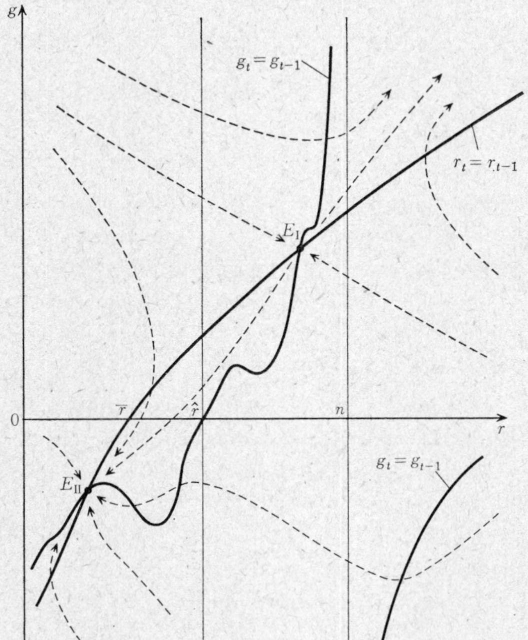

Figure 6

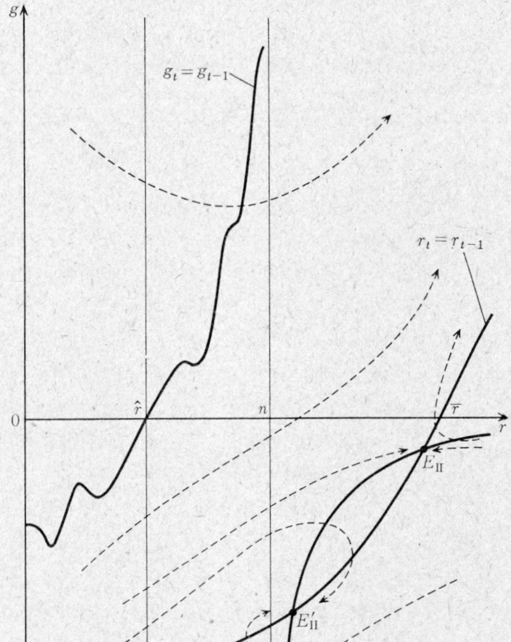

Figure 7

equilibrium is a saddle but type II equilibria may be locally unstable.

We now analyze the effect of a change of σ on the equilibrium rate of interest. By substituting (11) into (13) and by differentiating

$$\frac{dr}{d\sigma}=\frac{f''\left[\frac{\partial s}{\partial\sigma}+\frac{1}{r-n}\right]}{\alpha'\left[1-A'+\frac{B'E'}{r-n}\right]}$$

where

$$\alpha'=1+n-f''\frac{\partial s}{\partial\sigma},$$

$$A'=f''\frac{\partial s}{\partial y^d}(1-\tau)\phi'/\alpha',$$

$$B'=-f''/\alpha',$$

$$C'=A'E',$$

$$D'=B'E'+\frac{1+r}{1+n},$$

$$E'=\frac{g}{1+n}-\tau\phi'=\tau\phi\left[\frac{1}{r-n}-\frac{\phi'}{\phi}\right]-\frac{\sigma}{(r-n)(1+n)}.$$

Therefore, when $r>n$ and $g>0$, $\frac{dr}{d\sigma}<0$. On the other hand, when $r<n$, the sign of $\frac{dr}{d\sigma}$ cannot be determined *a priori*. However, when the equilibrium is locally stable[8], $\left[(1-A')\frac{n-r}{1+n}-B'E'\right]>0$ and hence the denominator of $\frac{dr}{d\sigma}$ is positive. The numerator can be written, after some calculation, as

$$\frac{-f''}{n-r}\left[1-\frac{\partial s}{\partial y^d}+\frac{1-n}{1+\sigma}\frac{\partial s}{\partial y^d}\right]$$

which is positive if $n<1$. Hence $\frac{dr}{d\sigma}>0$ when $r<n<1$.

This result can be interpreted as follows. An increase in σ, social security payment, will generate two effects on capital accumulation. First, an increase in social security will increase the lifetime income of consumers and thus increase saving or accelerate capital accumulation. Second, an increase in social security causes an incresase in government expenditure which will increase the government borrowing and hence suppress the private capital accumulation.

The ultimate impact of an increase in σ depends upon the relative magnitude of these opposite effects. The result above shows that, at a stable equilibrium, second effects usually dominates the first effects.

8) By Taylor expansion, (9) and (10) become

$$\begin{bmatrix} d\hat{r}_{t+1} \\ d\hat{g}_{t+1} \end{bmatrix}=\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}\begin{bmatrix} d\hat{r}_t \\ d\hat{g}_t \end{bmatrix}$$

where, as before, $d\hat{r}_t=r_t-r^*$, $d\hat{g}_t=g_t-g^*$, and (r^*, g^*) is an equilibrium. The local stability of (r^*, g^*) holds if and only if

$$(1+A')\left(1+\frac{1+r}{1+n}\right)<B'E'<(1-A')\frac{n-r}{1+n}.$$

5. Adaptive Expectation

The assumption of perfect foresight, which we have assumed in the previous sections, is rather extreme. In this section, in order to investigate the robustness of our model, we analyze the global property of an economy without social security program (σ) when expectation of wage in the next period is formed adaptively. Namely, let w_t^e be the expected level of wage prevailing in period t (which is formed in period $t-1$). Assume that

(14) $w_{t+1}{}^e=\beta w_t+(1-\beta)w_t^e$

for some $(0<\beta<1)$. Since firms expect wage rate in period $t+1$ to be $w_{t+1}{}^e$, under constant returns technology, firms will demand any amount of capital in period t at interest rate $\phi^{-1}(w_{t+1}{}^e)$.

With the same expectation, and with income $(1-\tau)w_t=(1-\tau)(f(k_t)-k_tf'(k_t))$ in period t, consumers save

$s((1-\tau)(f(k_t)-k_tf'(k_t)),\phi^{-1}(w_{t+1}{}^e))$

Therefore, capital labor ratio in $t+1$ will be

$$(15)\quad k_{t+1}=\frac{s((1-\tau)(f(k_t)-k_tf'(k_t)),\phi^{-1}(w_{t+1}{}^e))-g_t}{1+n}.$$

Finally, net accumulation of government debt per capita is described as

$$(16)\quad g_{t+1}-g_t=\frac{\phi^{-1}(w_{t+1}{}^e)-n}{1+n}g_t-\tau[f(k_{t+1})-k_{t+1}f'(k_{t+1})].$$

At an equilibrium of system (14)-(16), it is clear that $w_t=w_t^e$ and $\phi^{-1}(w_t^e)=f'(k_t)$. Therefore, the set of equilibria of the system (14)-(16) coincides with that of section 2. Hence, in general, there are two types of equilibria. Type I equilibria (E_{I}) have positive g and $r>n$, while type II equilibria (E_{II}) have negative g and $r<n$.

The system (14)-(16) is difficult to analyze, especialy from the global viewpoint, for the phase diagram must be three dimensional. Therefore, we shall confine ourselves within some observations directly derivable from equations (14)-(16).

With adaptive expectation, it remains to hold that too much debt issue will create crowding out and will increase both government debt outstanding and the rate of interest to catastrophic levels. This can be seen by the following observations. When government issues too much debt, capital labor ratio will decrease substantially (see (15)) because debt financing will crowd out private capital given expected rate of wage prevailing in the next period.

This decrease in capital labor ratio will decrease wage income as $\frac{dw}{dk}=-kf''>0$ and, at the same time, expected wage level will decrease. Hence the market rate of interest will increase as $\frac{d\phi^{-1}}{dw^e}=\frac{1}{\phi'}<0$. These two changes, i. e., decrease in wage income (and hence tax revenue) and increase in rate of interest, will force government to issue more debt than before. By this further increase in debt issue, the same process will repeat itself and consequently debt outstanding will increase at the expence of private capital decumulation. The economy must approach to a catastrophy unless drastic measures such as tax increase or discretionary inflation is taken.

On the other hand, if the original debt issue is whithin a reasonable range, an issue of government debt will still allow private capital to accumulate. Therefore, in the next period táx revenue will increase and the rate of interest will decrease, thus easing burden of debt. The economy will eventually approach to one of the equilibria described above.

6. Conclusions

In the paper, we analyzed the problem of the burden of debt in a simple dynamic economy. Contrary to the preceeding literature, we showed that an exceedingly large issue of government debt will eventually cause catastrophic burden to future generations. In this sense, government is recommended to use fiscal policy with care.

Needless to say, the positive tax rate is not likely to be maintained when the amount of debt outstanding becomes negative. Therefore, it is probably unlikely to happen that the economy will settle at a stable equilibrium if $g^* < 0$. It is rather the difficulty of achieving an equilibrium with $g^* > 0$ and $r^* \leqq n$ that is implicated by our analysis. As is well known (see, e. g., Benveniste [2], Okuno and Zilcha [8]) that equilibria with $r^* \geqq n$ are Pareto efficient and equilibria with $r^* < n$ are Pareto inefficient. Especially when $\tau > 0$, type I equilibrium with $r^* \geqq n$ is efficient. In this regard, it is likely that government policy may be aimed to achieve an equilibrium with $r^* \geqq n$ in the long run. Our analysis indicates that, in such a case, a discretionary fiscal policy (such as tax rate change) may be needed to keep the economy on the desired path.

Our model is, in many respects, too simple to give account of many important aspects of public debt problem. In particular, the lack of monetary consideration is inexcusable in that the burden of debt, if exists, is most likely to be solved by inflation. We will leave this problem as well as a more adequate formulation of capital and investment to other occasions[9].

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9) Okuno [9] attempts to answer this question, although the model is very restrictive.