Non-homothetic Production Functions Subject to Different Types of Technical Progress*

——Theory and Estimation——

Ryuzo Sato and Wilhelm Krelle

1. Introduction

The concept of an aggregate production function has proved to be a very fruitful one in economic theory, although there are those who would challenge this assertion. This concept is somewhat like a myth. Of course, there is no single output in the economy, and as a rule there are thousands of factors of production to be combined in a very complicated way, not likely to be described by a single equation called a "production function." Aggregation of different factors is subject to very restrictive conditions which will not be fulfilled in reality, and technical progress changes the production function (if it exists) in a way difficult to predict. Nevertheless, if one looks to an economy or a sector from a vantage point far enough away so that only some rough features may be distinguished and if one is content with a relation holding approximately true only for a limited number of periods, the production function is a useful tool in economic theory.

Under these circumstances, the properties of the production function have to be postulated and if an econometric estimation is to be done, the analytical form of the function has to be specified also. Since technical progress is an overwhelming fact in the production process in modern times, it is necessary to somehow incorporate it into the production function. If there were no rules regarding how the production relations systematically change as a result of technical progress, the concept of a production function would be of little analytical use. Most economists agree (and econometric tests support it) that there are certain relationships among certain economic variables which are invariant with respect to changes in technology (Arrow, *et. al.* [1], Jorgenson and Griliches [5], Sato and Beckmann [13, 14], Krelle [6]). These invariant relationships define the concept of "neutrality" in technical progress. The most well-known examples of "neutral" technical progress are: Hicks neutrality, Harrod neutrality, Solow neutrality and Sato-Beckmann neutrality. Unfortunately, agreement has not been reached among economists as to which type of "neutrality" fits the facts best[1]. Perhaps different types of neutrality are appropriate for different sectors of the

* The work was supported by the U. S. National Science Foundation Grant No. GS-41436 and a German Research Foundation grant. The authors wish to acknowledge assistance in econometric estimations by Knut Kübler of the Institute for Social Economic Sciences, Bonn. They are also grateful to the Editor of this *Review*, Professor Y. Kurabayashi, and the referee for their useful comments on an earlier draft.

1) There are reasons to favor Harrod neutrality. If *total* labor appears as an argument in the production function of an economy, then only improvement of the labor occupied in research and development is responsible for technical progress in machines. Moreover, since neoclassical growth theory depends on Harrod neutrality and this theory seems to explain some important features of developed economies in a reasonable way, growth theory itself recommends Harrod neutrality.

economy. Moreover, as a rule, these types of neutrality have been applied to production functions subject to a constant returns to scale technology or a monotone transformation of it, called *homothetic production functions*. But for sectors of the economy and for firms having U–shaped average cost functions and for the purpose of cross–sectional studies, it is much more appropriate to assume nonhomothetic production functions; cf. [8, 14]. Since homothetic production functions are a special case of non–homothetic functions nothing is lost by starting with non–homothetic functions and then switching to homothetic functions if it is not rejected.

This paper follows an earlier paper of Sato and Beckmann [14]. First, general explicit or implicit non–homothetic production functions which are compatible with different types of neutrality in technical progress are derived. Then these functions are specified in order to conduct econometric estimation employing German industry data. The results are used to ascertain homotheticity or non–homotheticity of the production function and to determine the type of neutrality of technical progress contingent on the specification of the production function employed in the empirical analysis.

2. Non–homothetic Production Functions Subject to Hicks Neutral Technical Progress

a. Partial Hicks Neutrality

Let

$Y=f(x_1, \cdots, x_n, T)$

be an explicit production function, where Y= final product, $x_1, \cdots, x_n$= factors of production, T= state of technology. Partial Hicks neutrality in technical progress is defined as that type of technical change that leaves the marginal rate of substitution between any two factors of production invariant during the change of technology as long as the amounts of these factors and the final product remains unchanges:

(1) $\omega_{ij}=f^{ij}(x_i, x_j, Y), \quad 1\leqq i, j\leqq n,$

where ω_{ij}= the marginal rate of substitution between the i^{th} factor x_i and the j^{th} factor x_j. Under the cost minimization behavior and fixed prices the marginal rate of substitution, being the ratio of the marginal products of these factors, is equal to the factor price ratio. Thus,

$$(1') \quad \omega_{ii}=\frac{Y_i}{Y_j}=\frac{p_i}{p_j}=f^{ij}(x_i, x_j, Y), \qquad 1\leqq i, j\leqq n,$$

where Y_i and Y_j are the marginal products of x_i and x_j and p_i, p_j are the prices of these factors. An invariant relationship such as equation (1) or (1′) implies certain characteristics of the underlying production function which will be derived now.

Equation (1) or (1′) defines a set of partial differential equations of the first order quasi–linear type, i. e.,

$$(1'') \quad \frac{\partial Y}{\partial x_i}-f^{ij}(x_i, x_j, Y)\frac{\partial Y}{\partial x_j}=0, \qquad 1\leqq i, j\leqq n, \qquad i\neq j,$$

with appropriate differentiability conditions. The equation of the characteristic is:

$$(2) \quad \frac{dx_i}{1}=\frac{dx_j}{f^{ij}(x_i, x_j, Y)}=\frac{dY}{0}, \qquad 1\leqq i, j\leqq n.$$

Integrating these we obtain:

(3) $F^{ij}(x_i, x_j, Y) = C^{ij}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n, Y, T),$
$1 \leqq i, j \leqq n, \quad i \neq j,$

where the arguments of the function C^{ij} are all inputs except x_i and x_j, Y and technical progress index T. The choice of i and j is completely arbitrary so that we could choose any of the $n(n-1)$ differential equations of equation (1''), (of which only $n(n-1)/2$ are independent). Since either x_i or x_j again appears in the right-hand term in all solutions with respect to ω_{ik} or ω_{jk} equation (3) must take the form:

(3') $F^i(x_i, Y) + F^j(x_j, Y) = C^{ij}(x_1, \cdots, x_n, Y, T), \qquad 1 \leqq i, j \leqq n.$

Therefore, the solution (3') may be extended by considering all other pairs of factor inputs to:

(4) $F^1(x_1, Y) + F^2(x_2, Y) + \cdots + F^n(x_n, Y) = \phi(Y, T).$

Technical progress enters into the ϕ function because it is a variable in the production function not included in the marginal rate of substitution function. Hence, we have:

Theorem 1: *If technical progress is partial Hicks-neutral, i. e., if the marginal rate of substitution between x_i and x_j is invariant under the given values of x_i, x_j and Y, $(1 \leqq i, j \leqq n)$, then the underlying production function must have the form* (4). *It is an implicitly additive and* (*possibly*) *non-homothetic production function with partial Hicks-neutral technical progress.*

Note that the separability condition in the marginal rate of substitution function is *not* necessary to derive the *implicitly* additive type of production function (see Sato and Beckmann [14]).

Of course, function (4) contains the well-known "explicitly Hicks neutral" case as a special case. For instance, if $\phi(Y, T) = Y/T$ and $\frac{\partial F^i}{\partial Y} \equiv 0$ (explicitly additive), then (4) can be written as

(5) $$Y = T \sum_{i=1}^{n} F^i(x_i),$$

which is an explicitly additive standard Hicks neutral technical progress function. It is important to note, however, that Hicks neutrality is only *one* special case of the explicitly additive production function

$$\sum_{i=1}^{n} F^i(x_i) = \phi(Y, T),$$

for we can readily imagine production functions which are explicitly additive but not explicitly Hicks neutral. Consider, for example:

(6) $$\phi(Y, T) = T^\delta + T^\gamma \cdot Y + T^2 Y^2 = \sum_{i=1}^{n} F^i(x_i)$$

which implies that

(7) $$Y = \frac{-T^\delta \pm \sqrt{T^{2\gamma} - 4T^2[T^\delta - \sum F^i(x_i)]}}{2T^2} > 0,$$

which is obviously not Hicks neutral since T cannot be factored out to yield

$$Y = T \cdot \sum_{i=1}^{n} F^i(x_i).$$

In the case of partial Hicks neutral technical progress and non-homotheticity, the effect of changes in T on Y is not simply multiplicative, but it depends on the following expression:

$$(8)\quad \frac{\partial Y}{\partial T}=\frac{\psi_T(Y, T)}{\sum_{i=1}^{n} F_Y^i(x_i, Y)-\psi_Y(Y, T)}.$$

Thus the effect depends not only on the factor inputs x_i, but also on the levels of Y and of T itself.

b. Total Hicks Neutrality

This type of technical progress is defined as the marginal rate of substitution between x_i and x_j not being affected by technical change as long as the levels of all factor inputs remain constant at a given isoquant:

$$(9)\quad \omega_{ij}=\frac{p_i}{p_j}=\frac{Y_i}{Y_j}=f^{ij}(x_1, \cdots, x_n, Y),\ \ 1\leq i,\ \ j\leq n.$$

That is to say, $\frac{\omega_{ij}}{\partial T}\equiv 0$[2].

Solving this system of partial differential equations we get

$$(10)\quad F(x_1, \cdots, x_n, Y)=\psi(Y, T(t)).$$

Thus we have:

Theorem 2: *If technical progress is of the kind of total Hicks neutrality, i. e., if the marginal rate of substitution between* x_i *and* x_j, $(1\leq i, j\leq n)$, *is invariant under the given values of* $x_1, \cdots, x_n$ *and* Y, *then the underlying production function has form* (10). *It is an implicit and (possibly) non-homothetic production function subject to total Hicks neutrality.*

3. Non-Homothetic Production Functions Subject to Harrod (or Solow) Neutral Technical Progress

a. The Two-Factor Case

We first consider the production function with two factors such as capital K and labor L. Harrod neutrality is defined by assuming that the marginal product of capital stays constant under technical progress as long as the output-capital ratio is constant (Sato and Beckmann [13]).

al. Extended Harrod Neutrality

We start with a slightly generalized concept of Harrod neutrality, called extended Harrod neutrality. We define it to imply that there exists an invariant relationship between the marginal product of capital and the amounts of capital and output. That is to say, the relationship

$$(11)\quad \frac{\partial Y}{\partial K}=f(Y, K),\ \ (f \text{ not necessarily homogeneous of degree one}),$$

remains unchanged under technical progress. Equation (11) can be treated as an ordinary differential equation yielding the solution

$$(12)\quad G(Y, K)=C(L, T),$$

or

$$(12')\quad Y=F[K, C(L, T)].$$

This is a non-homothetic production function subject to extended Harrod neutral technical progress. We may call it a *labor-affecting* technical progress. A special case of the above

2) We owe this to the referee.

is when $C(L, T)$ appears as

(13) $C(L, T)=TL.$

This is the case of labor augmenting technical progress.

a2. Harrod Neutrality Proper

If the marginal product of capital depends on the capital-output ratio such that

(14) $$\frac{\partial Y}{\partial K}=f(Y/K),$$

then the resulting production function must take the form

(15) $G(Y, K)=C(L, T),$

where $G(Y, K)$ is a homogenous function of the first degree.

a3. Extended Solow Neutrality

In the same manner as above the concept of Solow neutral technical progress can be extended. By interchanging K with L the "non-homothetic Solow neutral" production function can be expressed as

(16) $G(Y, L)=C(K, T)$

or

(16′) $Y=F[L, C(K, T)].$

Now technical progress is *capital–affecting*.

a4. Solow Neutrality Proper

For the case of

(14′) $$\frac{\partial Y}{\partial L}=F(Y/L),$$

we simply have

(15′) $G(Y, L)=C(K, T(t)),$

where G is a homogeneous function of the first degree.

b. The n–Factor Case, $n>2$

The concept of Harrod (or Solow) neutrality may be generalized to the n factor case in several ways. Since in the n factor case precludes only two types of factors of production, we shall call this extension: Harrod–Solow neutrality.

b1. Extended Single Factor Harrod–Solow Neutrality

We define this type of technical progress by assuming that technical progress does not affect the marginal product of *one specific factor* i as long as the amounts of this factor and output remain constant:

(17) $$\frac{\partial Y}{\partial x_i}=f(x_i, Y), \text{ for only one } i.$$

The underlying production function is derived by the solution of (17) as:

(18) $G(x_i, Y)=C(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, T),$

or

(18′) $Y=F[x_i, C(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, T)].$

Thus, except for the i^{th} factor input, Y depends, through the C–function, on all other factor inputs and the level of technical progress T. (18′) is a non–homothetic production function with technical progress affecting all factors except i. We summarize the main results in the following:

Theorem 3: *If in the two-factor case technical progress is Harrod (Solow)-neutral, i. e., if the marginal product of capital (labor) is not affected by technical progress as long as:* (1) *the output level and capital (labor) remain constant (extended case) or* (2) *the capital (labor)-output ratio remains constant (proper case), the production functions* (12′) *and* (16′) *respectively are (possibly) non-homothetic with capital (labor) affecting technical progress. In the n-factor case* ($n>2$), *the assumption of an extended single factor Harrod–Solow neutrality, as defined in* (17), *yields a non-homothetic production function* (18′) *where technical progress affects all factors except the* i^{th}.

b2. Extended Several Factor Harrod–Solow Neutrality

This case is defined by the invariant relationship (17) up to k factors of production:

$$(19)\quad \frac{\partial Y}{\partial x_i}=f^i(x_i,\ Y)\ \text{ for } i=1,1\cdots,k:\ k\leq n.$$

If $k<n$, the underlying relationship between Y and x must be

$$(20)\quad \sum_{i=1}^{k} G^i(x_i,\ Y)=\sum_{s=k+1}^{n} C^s(x_{k+1},\ \cdots,\ x_n,\ T_s)$$

or

$$(20')\quad Y=F\Big[x_1,\ x_2,\ \cdots,\ x_k,\ \sum_{s=k+1}^{n} C^s(x_{k+1},\ \cdots,\ x_n,\ T_s)\Big].$$

This is a non-homothetic production function where technical progress affects all factors $k+1,\ \cdots,\ n$, but not factors $1,\ \cdots,\ k$. If in (19) $k=n$, then equation (29) becomes

$$(21)\quad \sum_{i=1}^{n} G^i(x_i,\ Y)=C(T(t))$$

which is a special case of a non-homothetic production function with partial Hicks neutrality discussed in (4).

b3. General Harrod–Solow Neutrality

This still more general case is defined by replacing x_i in (19) with $x=(x_1,\ \cdots,\ x_n)$:

$$(22)\quad \frac{\partial Y}{\partial x_i}=f^i(x,\ Y)\ \ i=1,\ \cdots,k\,;\ k\leq n.$$

Now the underlying production relationship must be

$$(23)\quad G(x,\ Y)=\sum_{s=k+1}^{n} C^s(x_{k+1},\ \cdots,\ x_n,\ T_s(t))$$

or

$$(23')\quad Y=F\Big[x,\ \sum_{s=k+1}^{n} C^s(x_{k+1},\ \cdots,\ x_n,\ T_s(t))\Big].$$

This is a non-homothetic production function which depends on all factors of production and where technical progress separately affects all factors $k+1$ to n.

To summarize we have:

Theorem 4: *If in the n-factor case* ($n>2$) *technical progress is of the kind of general Harrod–Solow neutrality as defined by* (22), *the resulting non-homothetic production function represented by* (23′) *depends on all factors of production and separately on technical progress affecting those factors whose marginal products are not constant under technical change, given that* x *and* Y *are held constant.*

4. Non-homothetic Production Functions Subject to Sato–Beckmann Neutral Technical Progress

a. The Two-Factor Case

We will first consider the case of two factor inputs, capital K and labor L. Sato–Beckmann neutral technical progress is defined by assuming that the elasticity of substitution between K and L is not affected by technical progress as long as the relative share ratio s does not change at a given isoquant. Thus the basic invariant relationship which is not affected by technical progress is:

$$(24) \quad \sigma=\frac{dk}{d\omega}\frac{\omega}{k}=f(s, Y)$$

where

$$\omega=\frac{Y_L}{Y_K}, \quad k=\frac{K}{L} \text{ and } s=\frac{Y_L\cdot L}{Y_K\cdot K}=\frac{\omega}{k}.$$

Solving this differential equation we have

$$(25) \quad G(k, \omega, Y)=C(Y, T)$$

where G is homogeneous of degree one with respect to k and ω, and a monotone with respect to Y. Writing ω as

$$(25') \quad \omega=g[k, Y, C(Y, T)]=\frac{Y_L}{Y_K},$$

and solving the above partial differential equation we obtain:

$$(26) \quad F[K, L, Y, C_1(Y, T_1)]=C_2(Y, T_2),$$

where F is homogeneous of degree one with respect to K and L and T_1 and T_2 are two technical progress terms. This is an implicit non–homothetic production function subject to the Sato-Beckmann type of technical progress. A special case of the above, when

$$\frac{\partial C_1}{\partial Y}=\frac{\partial F}{\partial Y}\equiv 0, \quad C_2=\frac{Y}{T_2(t)} \text{ and } F(K, L, C_1(T_1))=F[T_1\cdot K, L] \text{ is}$$

$$F[T_1\cdot K, L]=\frac{Y}{T_2} \text{ or}$$

$$(27) \quad F[T_1\cdot T_2\cdot K, T_2\cdot L]=F[AK, BL]=Y.$$

This is the case of Sato–Beckmann technical progress applied to a constant returns to scale economy, cf. [13].

b. The n–Factor Case: $n>2$

Now the invariant relationships are:

$$(28) \quad \sigma_{ij}=\frac{\partial \log x_i/x_j}{\partial \log \omega_{ij}}=f^{ij}\left(\frac{\omega_{ij}}{x_i/x_j}, Y\right), \quad 1\leq i, j\leq n.$$

Solving the above system, we have

$$(28') \quad G^{ij}(\omega_{ij}, x_i/x_j, Y)=C^{ij}(Y, T^{ij})$$

where G^{ij} is a homogeneous function of degree one with respect to ω_{ij} and x_i/x_j. Rewriting the above we obtain

$$(29) \quad \omega_{ij}=\omega_{ij}[x_i/x_j, Y, C^{ij}(Y, T^{ij})], \quad 1\leq i, j\leq n.$$

But for ω_{ij} we may write

$$(29') \quad \frac{Y_j}{Y_i}=\omega_{ij}.$$

Combining the solutions of the independent parts of the above partial differential equation system, we derive

(30) $F[x, Y, A^1(Y, T_1), \cdots, A^{n-1}(Y, T_{n-1})]=\sum_{i=1}^{n-1} B^i(Y, T_i)$,

or more explicitly

(30′) $\sum_{i=1}^{n-1} F^i[x_i, Y, A^i(Y, T_i)]+F^n(x_n, Y)=\sum_{i=1}^{n-1} B^i(Y, T_i)$.

We summarize the results in the following theorem:

Theorem 5: *If in the two-factor case technical progress is Sato-Beckmann neutral, i. e., if the elasticity of substitution between capital and labor is not affected by technical progress as long as the relative shares do not change at given isoquant, the productive relation must be of the form* (26), *an implicit non-homothetic production function where technical progress is described by two terms. In the n-factor case* ($n>2$) *the more complicated implicit non-homothetic production function* (30′) *results, where there are n*–1 *terms describing technical progress.*

5. Cost Function and Technical Progress: Indirect Neutrality

Another approach to the study of productive relations under technical change is to work with a *cost function* instead of a *production function*. In this case "neutrality" has to be defined in terms of the cost function. In this section we derive general expressions for cost functions assuming different types of neutralities defined in terms of underlying cost relationships. These types of neutralities are referred to as "indirect neutrality." We assume the cost function C to be a concave homogeneous function of the first degree with respect to p_i, the i^{th} factor price, i. e.,

(31) $C=C(p_1, \cdots, p_n, Y, T)$.

Again the underlying production function may be non-homothetic.

a. Indirect Hicks Neutrality

This type of neutrality will be defined by

(32) $\frac{\partial C}{\partial p_i}\Big/\frac{\partial C}{\partial p_j}=f^{ij}(p_1, \cdots, p_n, Y), 1\leq i, j\leq n,$

where f^{ij}=homogeneous of degree zero with respect to p. This corresponds to total Hicks neutrality as defined in (9). Integrating the independent part of the above partial differential equations, we immediately obtain

(33) $F(p_1, \cdots, p_n, Y)=C(Y, T)$,

where F is homogeneous of degree one in all prices. This is an implicit cost function and the counterpart of the direct case, cf. equation (10). It may be more convenient to define

(34) $\frac{p_i}{C(Y, T(t))}=y_i, \ 1\leq i\leq n,$

so that we obtain

(35) $F(y_1, \cdots, y_n, Y)=1$.

Thus, the "real" cost of production of Y is *uniformly* affected by technical progress such that $\frac{p_i}{C(Y, T)}$ will be reduced uniformly by technical progress–which is the equivalent of Hicks neutrality.

b. Indirect Harrod-Solow Neutrality

This type of indirect neutrality will be defined in accordance to (17) as

(36) $$\frac{\partial C}{\partial p_i}=x_i=x_i(p_1,\cdots,p_n,Y)\text{ for one } i,$$

to yield

(37) $$C=p_i x_i(p_1,\cdots,p_n,Y)+\sum_{j\neq 1} p_j x_j(p_1,\cdots,p_n,Y,T).$$

c. Indirect Sato–Beckmann Neutrality

This type of technical progress will result from the invariant relationship between the elasticity of substitution and the ratio of cost shares (cf. [13]):

(38) $$\delta_{ij}=\frac{\partial\log\frac{p_i}{p_j}}{\partial\log\frac{C_j}{C_i}}=f^{ij}\left(\frac{\frac{C_j}{C_i}}{\frac{p_i}{p_j}},Y\right),\quad 1\leq i,j\leq n,$$

which yields

(39) $$C=\sum_{i=1}^{n-1}F^i[p_i,Y,A^i(Y,T_i)]+F^n(x_n,Y)=\sum_{i=1}^{n-1}B^i(Y,p_i).$$

To summarize the results:

Theorem 6: *The cost function will be affected by technical progress in such a way that*

$$F\left[\frac{p_1}{C(Y,T)},\frac{p_2}{C(Y,T)},\cdots,\frac{p_n}{C(Y,T)},Y\right]=1,$$

if and only if it is indirect non-homothetic Hicks neutral. The Harrod–Solow technical progress will yield the form:

$$C=p_i x_i(p,Y)+\sum_{j\neq 1} p_j x_j(p,Y,T).$$

The Sato–Beckmann type of factor cost reducing technical progress is represented by

$$C=\sum_{i=j}^{n-1}F^i[p_i,Y,A^i(Y,T_i)]+F^n(x_n,Y)=\sum_{i=1}^{n-1}B^i(Y,p_i).$$

6. Testable Production and Cost Functions Subject to Different Types of Technical Progress

In order to test which type of production (cost) function and which type of technical progress fits the statistical data best, the analytical form of some implicit or explicit production and cost functions derived in the foregoing section must be specified. This is not possible without some degree of arbitrariness, or to put it otherwise, without some additional assumptions which are sometimes quite restrictive. Therefore, if homotheticity and/or a special kind of neutrality of technical progress is to be accepted or rejected, this is to apply only for the specifications chosen in this section. Tables A and B summarize the specific types of production and cost functions used for estimation.

7. Empirical Estimation

a. Statistical Data

The German economy (except the government sector and nonprofit organizations) is subdivided into 12 sectors[3]. For each sector yearly figures from 1954 to 1967 are taken from

3) The twelve sectors are (1) agriculture, forestry, fishing, (2) gas, electricity, water, coal mining, (3) chemicals, stone, clay, glass, (4) iron, steel, non-iron, metals, (5) machines, vehicles, (6) electrical machinery, sheet-iron, (7) wood, paper, leather, textiles, (8) food, beverages and tobacco, (9) buildings, (10) trade, (11) transportation and communications, (12) other services. Classification of production sectors in the EWG: cf. Statistisches Amt der europäischen Gemeinschaften: Methodologie der Gemeinschaften

Table A Testable Production Functions Subject to Different Types of Technical Progress

Hicks Neutrality	Harrod–Solow Neutrality
MRS $\frac{Y_i}{Y_j}=\frac{p_i}{p_j}=a_{ij}\frac{x_i^{b_i}Y^{c_i}}{x_j^{b_j}Y^{c_j}}$	$\frac{Y_i}{Y_j}=\frac{p_i}{p_j}=a_{ij}\frac{x_i^{b_i}Y^{c_i}}{x_j^{b_j}Y^{c_j}}\cdot T^{d_{ij}}$
Implicit Form $\sum_{i=1}^{n}\alpha_i x_i^{-\rho_i}Y^{-\delta_i}=\psi(Y,T)$*	$\alpha_i x_i^{-\rho_i}\cdot Y^{-\delta_i}=\sum_{j\neq i}^{n}\alpha_j x_j^{-\rho_j}\cdot Y^{-\delta_j}\cdot T^{-\beta_j}$
*See Barnett, Kopecky, and Sato [17], Sato and Beckmann [14], Sato [12].	
Econometric Tests	Econometric Tests
$i=1, 2, 3,$ L(labor), K(capital), M(raw material).	Same as Hicks neutrality.
$P_L=w, P_K=r, P_m=$price of M.	
(IA)	(IIA)
$\text{Log}(w/r)=\log a_1+b_1\log L+c_1\log K+d_1\log Y$	Same as (IA)$+f_1t$, $T=t$
(IB)	(IIB)
$\text{Log}(r/P_m)=\log a_2+b_2\log K+c_2\log M+d_2\log Y$	Same as (IB)$+f_2t$
(IC)	(IIC)
$\text{Log}(P_m/w)=\log a_3+b_3\log M+c_3\log L+d_3\log Y$	Same as (IC)$+f_3t$.
Form of ψ–function	Harrod Neutrality
(i) $Y^{\gamma}\cdot(a+bT)$ or $aY+bT+c$	$f_1=-\beta_L$ $f_2=\beta_M,f_3=-\beta_M+\beta_L, \beta_L, \beta_M>0$
(ii) $Y^{\gamma}\cdot e^{a+bT}$ or $ae^{\alpha Y+\beta T}$	$\sum f_i\equiv 0.$
(iii) $Y^{\gamma}\cdot aT^{\beta}$	Solow Neutrality
	$f_1=\beta_K, f_2=-\beta_K+\beta_M,f_3=-\beta_M, \beta_K, \beta_M>0$
	$\sum f_i\equiv 0.$

consistent estimations made for a disaggregated forecasting system of the German economy. The figures consist of: invested gross and net capital K(=Brutto–und Nettoanlagevermögen), total labor force L (in billion of working hours, =Arbeitsvolumen), employed persons A (in millions, =Unselbständige), total wage bill W (in billions of DM, =gesamte Lohnsumme), profits G (in billion DM, =Gewinne), secondary inputs (=Vorleistungen) of all sectors to sector i, $i=1, \cdots, 12$. From these sources an average wage rate w is estimated by $w=W/A$ and a profit rate $r=G/K^{\text{net}}$. The price levels of the products of each sector were also available. Since we had only 14 observations for each variable we could not consider all secondary inputs but selected for each sector as "material input" only one input which was the most important for that sector. Therefore, material input M is only a *part* of all secondary inputs.

b. Results

There are eight sets of equations I to VIII with three equations in each set. Estimations have been made by the OLS–method for each of the 12 sectors. Table C shows the results of the estimations. There are sectors (e. g., No. 2, energy and coal mining) where the data fit almost all types of production and cost functions under consideration, and there is one sector (No. 5, machines and vehicles) where the data do not fit any of the functions considered. Among the production functions properties (IA–C and IIA–C), the Harrod–Solow neutrality definitely is preferable to the concept of Hicks neutrality. For all sectors for which a Hicks neutral production function cannot be rejected, a Harrod–Solow neutral production function also cannot be rejected; and there are many more sectors for which only this type

der Input-Output-Tabellen 1965, Sonderreihe 1, 1970 (NACE–CLIO (1965)).

Table B Testable Cost Functions Subject to Different Types of Technical Progress

1. First Specification

Factor Ratio $=\frac{x_i}{x_j}=\frac{\frac{\partial C}{\partial P_i}}{\frac{\partial C}{\partial P_j}}=a_{ij}\left(\frac{p_i}{p_j}\right)^b\cdot Y^{c_{ij}}\cdot T^{d_{ij}}$

Cost Function $=\left[\sum_{i=1}^{n}\alpha_i p_i^{-\rho}\cdot Y^{-\delta_i}\cdot T^{-\beta_i}\right]^{-\frac{1}{\rho}}=\phi(Y,\phi,T)=C.$

Econometric Tests

Indirect Hicks Neutrality

(IIIA) $\log(L/K)=\log a_1+b_1\log(W/L)+d_1\log Y$

(IIIB) $\log(K/M)=\log a_2+b_2\log(r/P_m)+d_2\log Y$

(IIIC) $\log(M/L)=\log a_3+b_3\log(P_m/w)+d_3\log Y.$

$b=b_j=b_i\langle=\rangle\rho=\rho_i=\rho_j$

Indirect Harrod–Solow Neutrality

(IVA) $=\cdots$ (IIIA) $+f_1t$

(IVB) $=\cdots$ (IIIB) $+f_2t$

(IVC) $=\cdots$ (IIIC) $+f_3t.$

More restrictions on d_i's and a_i's ($d_i=0$ $\langle=\rangle$ Homogeneity of D1).

2. Second Specification

Factor Ratio $\frac{x_i}{x_j}=\frac{\frac{\partial C}{\partial P_i}}{\frac{\partial C}{\partial P_j}}=a_{ij}\left(\frac{P_i}{P_j}\right)^{b_{ij}}Y^{c_{ij}}T^{d_{ij}}$

Cost Function $=\frac{\sum_{i=1}^{n}a_{ij}P_i^{b_{ij}+1}P_j^{-b_{ij}}Y^{-\delta_i}T^{-\beta_i}}{G(P,Y,T)}=C$, G=homogeneous degree zero with respect to P_i's.

Econometric Tests

Hicks Neutrality

(VA) $\log(L/K)=\log a_1+b_1\log W+c_1\log r+d_1\log Y$

(VB) $\log(K/M)=\log a_2+b_2\log r+c_2\log P_m+d_2\log Y$

(VC) $\log(M/L)=\log a_3+b_3\log P_m+c_3\log w+d_3\log Y.$

Restrictions on the coefficients:

($d_i\equiv 0\langle=\rangle$ Homogeneity of D1)

Indirect Harrod–Solow Neutrality

(VIA) $=\cdots$ (VA) $+f_1t$

(VIB) $=\cdots$ (VB) $+f_2t$

(VIC) $=\cdots$ (VC) $+f_3t.$

3. Third Specification

Factor Demand Function $=x_i=a_i\prod_{j=1}^{n}\left(\frac{p_j}{p_i}\right)\cdot Y^{\delta_i}\cdot T^{\beta_i}$

Cost Function $=\sum_{i=1}^{n}a_i p_i\prod_{j=1}^{n}\left(\frac{p_j}{p_i}\right)^{\alpha_{ij}}\cdot Y^{\delta_i}T^{\beta_i}=B(Y,T)=C.$

Empirical Tests

Indirect Hicks Neutrality

(VIIA) $\log L=\log a_1+b_{11}\log(r/w)+b_{12}\log(p_M/w)+b_{13}\log(r/p_m)+d_1\log Y$

(VIIB) $\log K=\log a_2+b_{21}\log(w/r)+b_{22}\log(p_m/r)+b_{23}\log(w/p_m)+d_2\log Y$

(VIIC) $\log M=\log a_3+b_{31}\log(w/p_m)+b_{32}\log(r/p_m)+b_{33}\log(w/r)+d_3\log Y.$

Indirect Harrod–Solow Neutrality

(VIIIA) $=\cdots$ (VIIA) $+f_1t$

(VIIIB) $=\cdots$ (VIIB) $+f_2t$

(VIIIC) $=\cdots$ (VIIC) $+f_3t.$

Restrictions on the coefficients

$a_i, d_i, f_i>0.$

If $d_i=d_j$ $\langle=\rangle$ cost function is homothetic.

Table C Types of Production (Cost) Functions and Types of Technical Progress in Different Sectors

Type of Production or cost function of technical progress	No. of the sector for which the production function is valid	Production Function accepted (+) if the parameters a, b, c, d, f lie in confidence intervals such that the regularity conditions are fulfilled for some relations of the parameters confidence interval						
(1)	(2)	90%	95%	97.5%	CES[1)]	Non-CES[2)]	Harrod[3)]	Non-Homothetic[4)]**
IA-C	2, 7, 8	+	+	+	2, 7	8	2, 7	8
Hicks		−	+	+		Hicks=2, 7, 8		
Neutrality		−	−	+				
	all other sect.	−	−	−				
IIA-C	1, 2 12	+	+	+	2, 7	1, 3, 8, 12	1, 2, 7, 8	3, 12
Harrod-Solow	3, 7, 8	−	+	+				
Neutrality	, 11	−	−	+		Harrod[5)]=2, 3, 8		
	all other sect.	−	−	−		Solow[6)]=1, 7, 12		
IIIA-C	12	+	+	+				12*
Indirect Hicks		−	+	+		$*d_i \neq 0$		
Neutrality		−	−	+				
	all other sect.	−	−	−				
IVA-C		+	+	+				6, 9, 10, 11
Indirect Harrod	10	−	−	+				
Solow Neutrality		−	−	+				
	all other sect.	−	−	−				
VA-C	2, 4, 6, 10	+	+	+		2		4, 6, 7, 9, 10
Indirect Hicks-	7, 9	−	+	+				
Neutrality	3	−	−	+				
	all other sect.	−	−	−				
VIA-C	3, 6, 10	+	+	+		3, 9, 12		6, 7, 10
Indirect	7, 9, 12	−	+	+				
Harrod-Solow	2	−	−	+				
Neutrality	all other sect.	−	−	−				
VIIA-C	2, 8, 11	+	+	+		2, 4, 8, 11*		
Indirect	4	−	+	+		$*d_i \equiv d$		
Hicks	7	−	−	+				
Neutrality	all other sect.	−	−	+				
VIIIA-C	2, 11	+	+	+		2, 3, 4, 11		
Indirect	3, 4	−	+	+				
Harrod-Solow	7	−	−	+				
Neutrality	all other sect.	−	−	−				

1) Condition in IA-C or IIA-C: $b_1=b_2=b_3=-c_1=-c_2=-c_3$.

2) Condition (1) is violated.

3) Condition in (A-C or IIA-C: $d_1=d_2=d_3=0$.

4) Condition (3) in IA is violated.

5) Condition in IIA-C: $f_1+f_2+f_3=0$ and $f_1<0$, $f_2>0$.

6) Condition in IIA-C: $f_1+f_2+f_3=0$ and $f_1>0$, $f_3<0$.

** Different types of production (cost) functions at 95% confidence interval.

of production function fits the data. As to different types of cost functions (IIIA–C to VIIIA–C) there seems to be a slight advantage of the explaining power of the cost function (49) (VA–C and VIA–C), judged from the number of sectors for which the data fit these types of cost function. The main lesson to be learned from Table C is that there is not *one* approach which is better in all cases, at least not on this level of abstraction[4].

Table C also shows the properties of the production functions IA–C and IIA–C for those sectors for which this approach cannot be rejected. Again there is no uniformity among the sectors: CES, non–CES, homothetic and non–homothetic Hicks–, Harrod– and Solow neutral production functions are present.

Turning to the cost function approach we can see that there are homothetic and non–homothetic cost functions. For cost functions IIIA–C to IVA–C, the non–homothetic type fits the data in almost all cases where there is a fit at all; for cost functions VIIA–C and VIIIA–C homotheticity is always realized. For cost functions VIA–C, homotheticity and non–homotheticity are both present.

Reviewing the results in total one may say that, on the given level of aggregation and for the production and cost functions used in the empirical investigation, there is not *one* type of function which is uniformly better for all sectors of the (German) economy. As a rule there are different approaches which fit the data equally well. *The major conclusion is that non–CES and non–homothetic production and cost functions cannot be neglected, and that in many cases they definitely fit better than the usually applied homothetic CES functions.*

(Ryuzo Sato: Brown University, U. S. A.)
(Wilhelm Krelle: University of Bonn, Germany)

References

[1] Arrow, K. J., H. B. Chenery, B. Minhas and R. M. Solow, "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, August, 1961, pp. 225–250.

[2] Beckmann, M. J., R. Sato, and M. Schupack, "Alternative Approaches to the Estimation of Production Functions and of Technical Change," *International Economic Review*, 1972, pp. 33–52.

[3] DeMeester, J. C., "Relations entre Coefficients de Fabrication et Fonctions de Production," Faculté des Sciences Economique et Sociales, Namur, Centre de Recherches, Discussion Paper 9, 1968.

[4] Houthakker, H. S., "Additive Preferences," *Econometrica*, Vol. 28: 2, (1960), pp. 244–257.

[5] Jorgenson, D. W. and Z. Griliches, "The Explanation of Productivity Change," *Review of Economic Studies*, July, 1967, pp. 249–283.

[6] Krelle, W., *Produktionstheorie*, 2. Aufl., Mohr, Tübingen, 1969.

[6a] Krelle, W., "Ersetzung der Produktionsfunktion durch preisund Kapazitätsabhängige Produktionskoeffizienten," Jahrbücher für Nationalökonomie und Statistik, Bd. 176 (1964), pp. 289–318.

[7] McFadden, D., "Constant Elasticity of Substitution Production Functions," *Review of Economic Studies*, April, 1963, pp. 73–77.

[8] Osborne, T. J., "Factor Substitution and Scale Effects," Unpublished Ph. D. Dissertation, Brown University, June, 1974.

---

4) The authors are aware of the defects of the test applied in this paper. A more systematic approach to statistical tests for selecting alternative functional forms and types of neutral technical proress may be adopted. One problem arising from the present test is that these two aspects are not completely separated. It is feasible to carry out a test between homothetic and non-homothetic forms and a test of various concepts of neutrality simultaneously. Here, a more appropriate test criteria may be F–values. We owe this to the referee. However, unfortunately, we were unable to follow his suggestions because the computer programs had already been erased.

[9] Paelinck, J., "Theoretische Eigenshaften der Produktionsfunktion," Institute für Statistik und Ökonometrie Universität Mannheim Discussion Paper No. 19, 1969.

[10] Samuelson, P., "Using Full Duality to Show that Simultaneously Additive Direct and Indirect Utilities Implies Unitary Price Elasticity of Demand," *Econometrica*, Vol. 33: 4 (1965), pp. 781–796.

[11] Sato, R., "On the Separable Class of Non-Homothetic CES Functions," *Economic Studies Quarterly*, April, 1974.

[12] Sato, R., "On the Most General Class of CES Functions," Mimeo., Brown University, March, 1973, *Econometrica*, July–Sept., 1975.

[13] Sato, R. and M. J. Beckmann, "Neutral Inventions and the Production Function," *Review of Economic Studies*, January, 1968, and January, 1969.

[14] Sato, R. and M. J. Beckmann, "Non-Homothetic Production Functions and Technical Progress," Paper presented at the Reisenberg Symposium on Productivities, June, 1974.

[15] Sato, R. and K. J. Kopecky, "'Implicit' Addilog Utility Functions," Mimeo., Brown University, March, 1974.

[16] Wolkowitz, B., "Homothetic Production Functions: A Theoretical and Empirical Analysis," unpublished Ph. D. Dissertation, Brown University, June, 1970.

[17] Barnett, W., K. J. Kopecky, and R. Sato, "Full-Information Maximum Likelihood Estimation of the Direct Implicit Addilog Demand Model," presented at the 1977 Econometric Society Meetings in New York.