EXISTENCE AND UNIQUENESS OF A SYSTEM OF CONSISTENT INDEX NUMBERS*

D. S. PRASADA RAO

In this paper, existence and uniqueness of a new system of consistent index numbers–for multilateral price and quantity comparisons–which is a refinement of the Geary (1958)–Khamis (1970) system, has been proved. The Geary–Khamis system of index numbers is based on a heuristic approach to the concept of 'general purchasing power.' The index numbers are obtained from the solution of a system of linear equations derived from the concept of 'general purchasing power.' This system of linear equations was shown to have solutions which are positive and unique, under very mild conditions to be satisfied by the price and quantity data [see Prasada Rao (1971)].

In this paper, the definitions involved in the Geary–Khamis system are modified to take into account some properties of economic theoretic price index numbers. In fact, the Geary–Khamis system can be shown to be a particular case of the more general system considered here. The relevant index numbers in the present system are shown to emerge from a system of non–linear equations. We show the existence and uniqueness of solutions to this system of non–linear equations, thus establishing the possibility of generalizing the Geary–Khamis system unambiguously. This, incidentally, establishes the possibility of using general theoretical concepts, used in binary comparisons, for multilateral price and quantity comparisons.

Some Concepts and Definitions

Throughout this paper, we consider only the comparison of price and quantity vectors of consumer goods. This simplifies the analysis considerably and provides a clear understanding of the method. Let $\boldsymbol{p}_j$ and $\boldsymbol{q}_j (j=1, \cdots, M)$ denote N–commodity price and quantity column vectors where typical elements p_{ij}, q_{ij} of these vectors denote the price or quantity consumed of the i th commodity in the j th vector. These M vectors of price and quantity may correspond to different time periods or different situations at a particular point of time.

The Geary–Khamis system of index numbers hinges mainly on the differences in the 'general purchasing power' of money due to differences in prices. This method tries to obtain relative magnitudes of the 'general purchasing power' of money as absolute values of general purchasing

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power are meaningless in the present context. In fact they make use of the reciprocals of the general purchasing power which they call the 'exchange rate.' This new concept allows one to calculate an 'average price' level for each commodity after making allowance for different purchasing powers corresponding to different price vectors. Conceptually, it is meaningless to add absolute prices of a commodity in different situations since the purchasing power of money is different in all these situations. This point becomes more apparent when one considers $\boldsymbol{p}_j's$ to be price vectors in different currencies.

Let R_j represent the 'exchange rate' or the reciprocal of 'general purchasing power of money in the j th situation, for each j, and P_i denote the average price of the i th commodity, averaged over different price vectors after allowing for differences in purchasing powers. If I_{jk} denotes the price index number for the k th price vector with the j th price vector as base, then it is defined as R_j/R_k.

So far the approach has been fairly general and it remains to be seen how the observed price and quantity data can be made use of to quantify $R_j's$ and $P_i's$.

The Geary–Khamis system uses a set of equations to determine, simultaneously, $R_j's$ and $P_i's$. They are determined by a system of $M+N$ equations below.

For $i=1, \cdots, N$.

$$P_i=\frac{\sum_{j=1}^{M} R_j p_{ij} q_{ij}}{\sum_{j=1}^{M} q_{ij}} \tag{1a}$$

and for $j=1, \cdots, M$,

$$R_j=\frac{\sum_{i=1}^{N} P_i q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}} \tag{1b}$$

Briefly, the 'average price' P_i is determined by the total expenditure on the i th commodity, over all the M situations after converting each $p_{ij}q_{ij}$ into a common denomination through R_j and the total quantity of the i th commodity consumed over all situations. This definition is fairly obvious and intuitive. Then the exchange rate is defined as the ratio of expenditures $\sum_i P_i q_{ij}$, at average prices, and $\sum_i p_{ij} q_{ij}$ at actual price levels. Under very mild conditions on the $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$ vectors, exixtence and uniqueness of a positive solution for the $R_j's$ and hence the $P_i's$, was proved in Prasada Rao (1971). The Geary–Khamis system of index numbers is defined from the solution that emerges from equation system (1).

While the definition of 'average price' is intuitively obvious it can be seen that the definition of 'exchange rates' is only one of many ways of defining these values through the given data. In fact, falling back on the traditional economic theory of index numbers, one can define the $R_j's$ in a more meaningful way.

Let U_j represent the utility function corresponding to the j th situation and vectors $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$. We assume that, for each j, U_j is a well behaved utility function possessing properties like continuity used in demand analysis. Assuming that the individuals are utility maximizers and since $\boldsymbol{q}_j$ is a vector of quantites consumed with price vector $\boldsymbol{p}_j$, $\boldsymbol{q}_j$ can be seen to be maximizing $U_j(\boldsymbol{q})$ subject to the restriction $\boldsymbol{p}_j'\boldsymbol{q}=\boldsymbol{p}_j'\boldsymbol{q}_j$.

In this sitution, the exchange rate R_j should be defined on the basis of minimum expenditure necessary to attain $U_j(\boldsymbol{q}_j)$ with the 'average price' vector $\boldsymbol{P}=(P_1\cdots P_N)'$. Then the exchange rate can be defined as the ratio of this expenditure to the actual. Note that this is analogous to the theoretical 'true price index number' defined in the literature. Hence, for each j, we have

$$R_j=\frac{\text{Min}\{\boldsymbol{P}'\boldsymbol{q}\,|\,U_j(\boldsymbol{q})=U_j(\boldsymbol{q}_j)\}}{\boldsymbol{p}_j'\boldsymbol{q}_j} \tag{2}$$

It can be seen that this definition of R_j coincides with the definition in equation system (1) if each utility function U_j is of fixed-coefficient type, for then

$$\min\{\boldsymbol{P}'\boldsymbol{q}\,|\,U_j(\boldsymbol{q})=U_j(\boldsymbol{q}_j)\}=\sum_{i=1}^{N}P_i q_{ij}.$$

Further the vector of values of R_j for $j=1,\cdots,M$ in system (1) gives an upper bound to the R_j's defined above, or equivalently, the Geary–Khamis system overestimates the price changes.

By defining the exchange rates as above, we have generalized the Geary–Khamis system to take the utility functions into consideration explicitly. This approach also demonstrates the possibility of using the concept of 'true price or cost-of-living index number' for multilateral comparisons. The new index number system is based on the solution for vectors $\boldsymbol{r}^*=[R_1\cdots R_M]'$ and $\boldsymbol{P}=[P_1\cdots P_N]'$ from the following equation system.

For each i,

$$P_i=\frac{\sum_{j=1}^{M}R_j p_{ij} q_{ij}}{\sum_{j=1}^{M}q_{ij}} \tag{3a}$$

and for each j,

$$R_j=\frac{\text{Min}\{\boldsymbol{P}'\boldsymbol{q}\,|\,U_j(\boldsymbol{q})=U_j(\boldsymbol{q}_j)\}}{\sum_{i=1}^{N}p_{ij}q_{ij}} \tag{3b}$$

Our system can be a well defined system only if the equations (3a) and (3b) yield solutions which can be used for price comparisons. To show that this system is viable, it is sufficient if we show that the system of equations yields a positive and unique (up to multiplication by positive scalars) solution for the R_j's. We shall prove this result in the next section. Throughout this paper we make the following two assumptions with respect to the observed price and quantity vectors.

Assumption 1: For all i and j, $p_{ij}>0$.

Assumption 2: For all i and j, $q_{ij} > 0$.

These assumptions are restrictive, but they simplify the proofs considerably. In fact assumption 2, which is the more restrictive of the two, can be relaxed without affecting the results in the next section.

Existence and Uniqueness

Throughout this section, we shall focus attention on solving for the vector of exchange rates $\boldsymbol{r}^*$ in the system of equations (3a) and (3b). While direct substitution of the equations for the $P_i's$ into the equations for the $R_j's$ in (1) simplifies matters considerably in proving existence of the Geary–Khamis system of index numbers, a similar approach is of no avail in the present context.

Instead we view the equation system (3) as an iterative scheme to obtain the values of the $R_j's$, since any initial vector of values $\boldsymbol{r}_0^* = [R_1^0 \cdots R_M^0]$ will lead to a sequence of vectors $\boldsymbol{r}^*$. This kind of approach allows us to draw heavily from the literature on non–linear homogeneous difference equations.

Let R_M^+ denote the non–negative orthant of M–dimensional Euclidean space. We will define a function H, mapping R_M^+ into itself, using the system of equations (3). For any given $\boldsymbol{r}^* = (R_1 \cdots R_M) \in R_M^+$, we can define $P(\boldsymbol{r}^*)$ from equations (3a) and using the vector $P(\boldsymbol{r}^*)$, we can define $H(\boldsymbol{r}^*)$ from equations (3b).

Let us state a few conditions below and examine the properties possessed by the above class of functions. Consider a general mapping $F(\boldsymbol{r})$ such that:

(α) $F(\boldsymbol{r})$ maps R_M^+ into itself. Further $\boldsymbol{r} \geq 0 \rightarrow F(\boldsymbol{r}) > 0$. (1)

(β) Continuity: $F(\boldsymbol{r})$ is a continuous function, from R_M^+ into itself, at all points in R_M^+, except possibly at $\boldsymbol{r} = 0$.

(γ) Homogeneity: $F(\boldsymbol{r})$ is positively homogeneous of degree m, $1 \geq m \geq 0$ in the sense that

$$F(\lambda \boldsymbol{r}) = \lambda^m F(\boldsymbol{r}) \text{ for all } \lambda \geq 0 \text{ and } \boldsymbol{r} \geqq 0.$$

(δ_w) Weak Monotonicity: For all $\boldsymbol{r}, \boldsymbol{s} \in R_M^+$,

$$\boldsymbol{r} \geqq \boldsymbol{s} \rightarrow F_i(\boldsymbol{r}) \geq F_i(\boldsymbol{s}) \text{ for all } i \text{ such that } R_i = S_i$$

(δ) Monotonicity: For all $\boldsymbol{r}, \boldsymbol{s} \in R_M^+$,

$$\boldsymbol{r} \geqq \boldsymbol{s} \rightarrow F(\boldsymbol{r}) \geqq F(\boldsymbol{s})$$

(δ_s) Strict Monotonicity: For all $\boldsymbol{r}, \boldsymbol{s} \in R_M^+$,

$$\boldsymbol{r} \geq \boldsymbol{s} \rightarrow F(\boldsymbol{r}) > F(\boldsymbol{s}).$$

We now relate these conditions to the function H previously defined. Given Assumptions (1) and (2) and the fact that the utility functions U_j are continuous and well behaved, our function H, defined through equations (3a) and (3b), can be shown to satisfy conditions (α), (β),

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(1) Here we use the convention that for any $\boldsymbol{x}, \boldsymbol{y} \in R_M^+$,

$$\boldsymbol{x} \geqq \boldsymbol{y} \Leftrightarrow x_m \geq y_m \text{ for all } m$$

and $\boldsymbol{x} \geq \boldsymbol{y} \Leftrightarrow x_m \geq y_m$ for all m and $x_m > y_m$ for at least one m.

(γ), (δ_s) and hence (δ_w) and (δ). Further it can be seen that H is positively homogeneous with $m=1$.

As already observed, the function H defined through (3a) and (3b) can be used in an iterative scheme which essentially resembles a non–linear homogeneous difference equation of the type

$$\boldsymbol{x}(t+1)=H(\boldsymbol{x}(t)) \tag{4}$$

where the causal relation $\boldsymbol{x}(t)\rightarrow\boldsymbol{x}(t+1)$ is defined through a function H mapping R_M^+ into itself.

Definition:

A balanced growth solution $\boldsymbol{x}(t)$ of (4) is defined as a non–trivial solution such that the proportions $\boldsymbol{x}_1(t):\boldsymbol{x}_2(t):\cdots:\boldsymbol{x}_M(t)$ of the components of $\boldsymbol{x}(t)$ remain constant over time.

Since we are interested in the convergence of the solution in the sense that the index numbers defined must be the same after some steps, it is necessary to concentrate on the existence of a balanced growth solution to the equation system (4). Further we are interested in a balanced growth solution which is unique up to multiplication by positive scalars. In fact it is necessary to show that we arrive at the same set of index numbers irrespective of the initial point in the iterative scheme suggested in equations (3a), (3b) and (4).

We prove this by stating a few theorems from Nikaido (1968, pp. 150–161), without proofs.

If we are interested in the balanced growth solution to equation system (4), we need to consider the eigen–value problem

$$H(\boldsymbol{x})=\lambda\boldsymbol{x} \tag{5}$$

and its solutions. Then we have

Theorem 1:

Under conditions (α) and (β), the eigen value problem (5) is solvable for some $\lambda\geqq 0$.

Theorem 2:

Let H satisfy properties (α), (β) and (γ) and let

$$\Lambda=\{\lambda\,|\,H(\boldsymbol{x})=\lambda\boldsymbol{x} \text{ for some } \boldsymbol{x}\in P_M\}$$

where $P_M=\left\{\boldsymbol{x}\,|\,\boldsymbol{x}\in R_M^+ \text{ and } \sum_{i=1}^{M}x_i=1\right\}$ is the standard simplex. Then Λ contains a maximum which is denoted by $\lambda(H)$. Furthermore when H is positively homogeneous of degree 1, then $\lambda(H)$ is the largest among all eigen values of H.

Before we proceed further, let us introduce the concept of indecomposability. For any two vectors $\boldsymbol{x},\boldsymbol{y}\in R_M^+$, let

$$N(\boldsymbol{x},\boldsymbol{y})=\{j\,|\,x_j>y_j\}$$

Obviously $\qquad N(\boldsymbol{x},\boldsymbol{y})\subset\{1,2,\cdots,M\}$.

Definition:

$H(\boldsymbol{x})=(H_i(\boldsymbol{x}))$ is said to be indecomposable if for any pair of vectors $\boldsymbol{x},\boldsymbol{y}$ such that $\boldsymbol{x}\geqq\boldsymbol{y}\geqq 0$ and $N(\boldsymbol{x},\boldsymbol{y})$ is a proper subset of N, then $H_i(\boldsymbol{x})\neq H_i(\boldsymbol{y})$ for some $i\notin N(\boldsymbol{x},\boldsymbol{y})$. If H satisfies condition (δ_w), then

$$H_i(\boldsymbol{x})>H_i(\boldsymbol{y}) \text{ for some } i\notin N(\boldsymbol{x},\boldsymbol{y}).$$

Note: The function H defined through equations (3a) and (3b) satisfies (δ_S) and hence it is indecomposable. Indecomposability plays a very important role in proving the existence of the Geary–Khamis system [see Prasada Rao (1971)].

Theorem 3:

If (α), (β) and (δ_w) and indecomposability are assumed and $M \geq 2$, then

(i) The eigen values and eigen vectors of H are positive.

(ii) The eigen vectors are unique up to multiplication by positive scalars. Here the uniqueness is asserted over all possible eigen vectors arising in association with any eigen values of H.

Note: This theorem is useful in establishing uniqueness up to multiplication by positive scalars of the resulting soltions for the R_j's.

Definition:

The function $H(\boldsymbol{x}) = (H_i(\boldsymbol{x}))$ is said to be primitive at $\boldsymbol{x} = \boldsymbol{a} \geqq 0$ if for any $\boldsymbol{y}$ such that $\boldsymbol{y} \geq \boldsymbol{a}$, there is a positive integer for which $H^s(\boldsymbol{y}) > H^s(\boldsymbol{a})$.

Note: When H satisfies (δ_s) then $H(\boldsymbol{x})$ is primitive at all points $\boldsymbol{a} \in R_M^+$ and further this happens for $s=1$. Hence the function H defined through equations (3a) and (3b) is primitive at all points in R_M^+.

Since the function H is positively homogeneous of degree 1 and also satisfies (α), (β), (γ) and (δ_s) (and hence primitivity at all points in R_M^+), we have $\lambda(H) > 0$ and the difference equation $\boldsymbol{x}(t+1) = H(\boldsymbol{x}(t))$ has a balanced growth solution

$$\boldsymbol{u}(t) = \lambda^t \boldsymbol{u}$$

where $\boldsymbol{u}$ is a positive eigen vector associated with $\lambda = \lambda(H)$. Further $\boldsymbol{u}$ is unique up to multiplication by positive scalars. This follows from Theorem (3) stated above.

The discussion above ensures that there exists a balanced growth solution, for the unknown vector of exchange rates defined in (3a) and (3b) which possesses all the required properties. The following theorem from Nikaido establishes the fact that irrespective of the initial solution one may start with, the iterative scheme results in a solution which converges in ratio to the balanced growth solution.

Theorem 4:

If the function H in equation (4) satisfies conditions (α), (β), (δ), positive homogeneity of degree one, (δ), indecomposability and primitivity at points 0 and $\boldsymbol{u}$, then every solution $\boldsymbol{x}(t) = (\boldsymbol{x}_i(t))$ of (4), starting at initial point $\boldsymbol{x}(0) \geq 0$, converges in ratio to the balanced–growth solution, i. e.,

$$\lim_{t\to\infty} \frac{x_i(t)}{u_i(t)} = \gamma \text{ for } i = 1, \cdots, M.$$

the limit being the same for all i.

This theorem establishes the fact that equation system (3a) and (3b) will lead to the same price index numbers (which are ratios of the R_j's), whatever may be the initial point used in the iterative procedure having form (4). The fact that $\lim_{t\to\infty} x_i(t)/u_i(t) = \gamma$ is the same for all i, is crucial. Since the balancd growth solution u is itself both unique up to multiplication by positive scalars and strictly positive this ensures the existence and uniqueness of the price index numbers defined through (3a) and (3b).

The discussion above establishes the following result.

Main Theorem:

Under Assumptions (1) and (2), and if U_j is continuous and well-behaved for each j then the equation system (3a) and (3b) yields meaningful price and hence quantity comparisons.

The result above states that the new index number system is viable in spite of the non-linearities involved. One can still derive price and quantity index numbers which are unambiguous. This demonstrates the possibility of generalizing the Geary–Khamis system and establishes the fact that the concept of 'true cost-of-living index number' can be used in multilateral comparisons which are consistent.

There are two areas of further research which the author is presently investigating. Firstly, the assumptions (1) and (2) can be weakened without affecting the main result which depends on much weaker conditions like (δ), (δ_s), indecomposability of H and primitivity of H at different points. It would therefore be very interesting to derive some necessary and sufficient conditions for the main result in this paper to hold, in terms of price and quantity data. Secondly, the final price and quantity index numbers obtained through iteration on (3a) and (3b) depend explicitly on the particular functional form of the U_j's. There are two separate problems here. One is the problem of specification of the utility functions while the other is concerned with finding a balanced growth solution explicitly.

Until the second problem is resolved satisfactorily, the main result of this paper will remain as an existence theorem, showing the existence of the possibility of using the concept of 'true cost-of-living index numbers' for consistent multilateral comparisons.

(University of New England, Armidale, Australia)

References

[1] Geary, R. C. (1958), "A note on the comparison of exchange rates and purchasing powers between countries," *Journal of the Royal Statistical Society* (Series A), Vol. 121, 97–99.

[2] Khamis, S. H. (1970), "Properties and conditions for the existence of a new type of index numbers," *Sankhyā*, (Series B), Vol. 32, 81–98.

[3] Nikaido, H. (1968), *Convex Structures and Economic Theory*, New York, Academic Press.

[4] Prasada Rao, D. S. (1971), "Existence and uniqueness of a new class of index numbers," *Sankhyā*, (Series B), Vol. 33, 341–354.