On Acyclic Preferences

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1. Introduction

The pure theory of preference and rational choice investigates the relationship between preference relations and choice functions. In abstraction from internal structure of the alternatives of choice, it studies, on the one hand, the rationalizability of the given choice function in terms of an appropriate preference relation and, on the other, the possibility of generating a choice function from the given preference relation. (Arrow [1], Herzberger [3], Richter [2, Chapter 2], Sen [7], and others.) One of the conventional wisdom of the second part thereof tells us that *a connected and acyclic preference relation can generate a choice function over a family of finite sets.* (Herzberger [3, p. 202 & p. 206] and Sen [6, p. 16]) It is also well-known that *a preference relation, defined over a topological space, satisfying the ordering axiom of connectedness and transitivity can derive a choice function over a family of compact sets if it is upper semi-continuous.* (Nikaido [4, p. 240]) Synthesizing these results, we will prove a fairly general theorem on generating a choice function from the preference relation. Theorem will be proved in section 2, while some clarifying comments will be given in section 3.

2. Theorem

2.1 Let X be the basic set of alternatives and let K stand for the non-empty family of non-empty subsets of X. (Each and every $S \in K$ is construed to be the set of available alternatives which could possibly be presented to the agent.) A *preference relation* R over X is a binary relation defined on X such that xRy for x, y in X means that x is at least as good as y. Associated with R, *strict preference relation* P and *indifference relation* I are defined by xPy if and only if [xRy & not (yRx)] and xIy if and only if [xRy & yRx], respectively. A *choice function* C over K is a function which associates a non-empty subset $C(S)$ of S to each and every $S \in K$. Choice function C is said to be *closed* if $C(S) \in K$ for all $S \in K$ (Smith [8, p. 186]) and *regular* if (i) K includes all singletons taken from X, and (ii) $S_1, S_2 \in K$ implies $S_1 \cup S_2 \in K$ (Herzberger [3, p. 191]).

2.2 Preference relation R generates two natural candidates for choice functions. Firstly we have the set of R-maximal elements:

(1) $M_R(S) = \{x^* : x^* \in S \text{ \& not } (yPx^*) \text{ for all } y \in S\}$.

Secondly we have the set of all R–greatest elements:

(2) $G_R(S) = \{x^* : x^* \in S \text{ \& } (x^*Rx) \text{ for all } x \in S\}$.

In order for M_R and G_R to be well-defined choice functions over K, we must have $M_R(S) \neq \phi$ and $G_R(S) \neq \phi$ for all $S \in K$. Reasonable and widely applicable restrictions on R and K to that effect will be given in what follows. It should be noticed in this context that (i) any two elements of $G_R(S)$ are R–indifferent, while any two elements of $M_R(S)$ are either R–indifferent or R–incomparable, and (ii) $G_R(S) \subset M_R(S)$ holds generally for all $S \subset X$ and they coincide if R is *connected* (xRy and/or yRx for all x, y in X). In view of (ii), we have only to consider the workability of M_R as a choice mechanism. Property of G_R can be derived therefrom if we impose the connectedness axiom additionally.

2.3 We assume that X is a topological space with an appropriate collective family of open sets. We also assume that K is the family of non-empty compact sets in X, that is to say, for any $S \in K$, (i) $S \neq \phi$, and (ii) every family of open sets whose union covers S has a finite sub-family whose union covers S. Now let $L(x)$ stand for the set of all alternatives which are preferentially dominated by x: $L(x) = \{y : y \in X \text{ \& } xPy\}$. We say that R is *upper semi-continuous*——(C), for short——if $L(x)$ is an open set for each and every x in X. We say that R is *acyclic*——(A), for short——if there exists no subset $\{x^1, x^2, \cdots, x^t\}$ of X such that $x^1Px^2P\cdots Px^tPx^1$ for all $t=1, 2, \cdots$, ad inf. Now the theorem.

Theorem: Under (A) and (C), M_R is a closed and regular choice function over K.

2.4 Proof of well-definedness goes as follows. Let $S \in K$ be fixed once and for all. For any x in X, we define $V(x, S)=S\cap L(x)$. Thanks to (C), $V(x, S)$ is a relative open set in S, so that $\{S-V(x, S): x\in S\}$ is a family of relative closed sets in S. We have

(3) $M_R(S)=\bigcap_{x\in S}\{S-V(x, S)\}$.

Suppose that $M_R(S)=\phi$. Noticing that $S-\bigcup_{x\in S}V(x, S)=\bigcap_{x\in S}\{S-V(x, S)\}$, the family of relative open sets in S, $\{V(x, S): x\in S\}$, is seen to form an open covering of S. S being compact, there exists a finite sub-family $\{V(x^k, S): x^k\in S\ (k=1, 2, \cdots, n)\}$ satisfying

(4) $S=\bigcup_{k=1}^{n}V(x^k, S)$.

Now let $y^0\in S$ be fixed once and for all. Because $M_R(S)=\phi$, there exists a $y^1\in S$ such that y^1Py^0. Thanks to (4), there exists a $z^1\in\{x^1, x^2, \cdots, x^n\}$ satisfying z^1Py^1. Corresponding to this z^1, there exists a $y^2\in S$ such that y^2Pz^1. Repeating this procedure, we obtain two infinite sequences $(z^\mu)_{\mu=1}^\infty$ and $(y^\mu)_{\mu=1}^\infty$ in S such that

(5) $z^\mu\in\{x^1, x^2, \cdots, x^n\}$ ($\mu=1, 2, \cdots$ ad inf.)

(6) $y^{\mu+1}Pz^\mu Py^\mu$ ($\mu=1, 2, \cdots$ ad inf.)

If $z^{\mu^*}=z^{\mu^{**}}$ for some $\mu^*<\mu^{**}$, we have $z^{\mu^*}Py^{\mu^*}Pz^{\mu^*-1}P\cdots Py^{\mu^{**}+1}Py^{\mu^{**}}$ by virtue of (6), in contradiction to (A). Thus every term in $(z^\mu)_{\mu=1}^\infty$ should be distinct, which is impossible because of (5). By *reductio ad absurdum*, we have $M_R(S)\neq\phi$. Closedness follows from (3). Finally regularity is trivial, because (i) any singleton set in X is compact, and (ii) the union of a finite number of compact sets is compact.

3. Concluding Remarks

As is well known, if R is *transitive* (xRy and yRz imply xRz for all x, y and z in X), both P and I are transitive, while the transitivity of P implies acyclicity of R. (Sen [6, p. 10 & p. 16]) It might be doubted that the combination of upper semi-continuity and acyclicity is so strong that, under upper semi-continuity, acyclicity could be equivalent to transitivity. In order to dissipate this doubt, some examples showing that R can be acyclic without being transitive even under upper semi-continuity will be given.

Example 1. Let X be the set of all real numbers (with usual topology) and let R be defined by [xRy if and only if $y\leqq x<y+1$] for all x, y in X. Noticing that [xPy if and only if $y<x<y+1$] for all x, y in X, this R is easily seen to be acyclic. But the transitivity of R does not necessarily hold. For any x in X, let $y\in L(x)$. For such an ε as to satisfy $0<\varepsilon<\min\{x-y, y+1-x\}$, we have $]y-\varepsilon, y+\varepsilon[\subset L(x)$, so that R is upper semi-continuous.

Example 2. Let $X=\{x^1, x^2, x^3\}$ and let the discrete topology be introduced in X. (Any subset of X is then an open set.) Let R be defined by $x^1Px^2Px^3$, x^1Ix^3. This R is connected and acyclic but not transitive. It is also upper semi-continuous with respect to discrete topology.

Thanks to these examples, we can claim that our theorem is a genuine generalization of the second traditional proposition referred to in section 1. In order to see that the first traditional proposition is also contained in our theorem, we have only to introduce discrete topology into the set of all alternatives.

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