GLOBAL UNIVALENCE AND STABILITY OF COMPETITIVE MARKETS*

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In recent years considerable attention has been paid to the question of global univalence for economic transformations, e. g., Gale and Nikaido [1] and Nikaido [6]. However, very little has been said regarding the relationship between univalence and economic stability. The purpose of this paper is to fill this gap. We shall study the relationship between the conditions of global univalence and of global stability of competitive equilibrium. Since both the conditions of global univalence and global stability largely depend on some properties of the Jacobian matrix of a transformation, these two problems are closely related. Consider, for instance, a simple excess demand function in an isolated market:

$$\dot{p}=f(p).$$

Under certain conditions we can infer that if there exists an equilibrium point $\dot{p}=0$ for $p=p_0$, then the transformation $f(p)$ is globally one-to-one and each solution of $f(p)=0$ is globally stable. That is to say, if the derivative of $f(p)$ is negative for all p, then the transformation is globally one-to-one and the equilibrium point is globally stable (see Figure 1). Here, we note that a sufficient condition for global univalence is also a sufficient condition for global stability. It is obvious, however, that global univalence is not necessary for global stability, as Figure 2 shows. In Figure 2, the transformation is not one-to-one for all p, but the equilibrium point is globally stable.

Figure 1

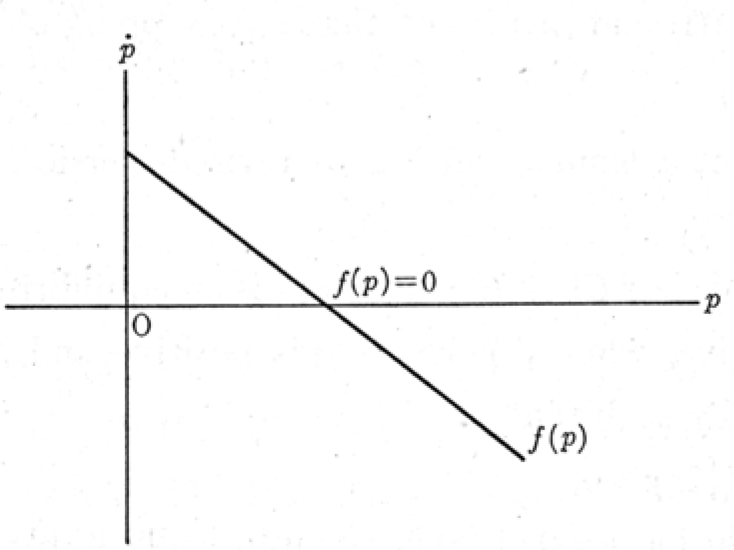
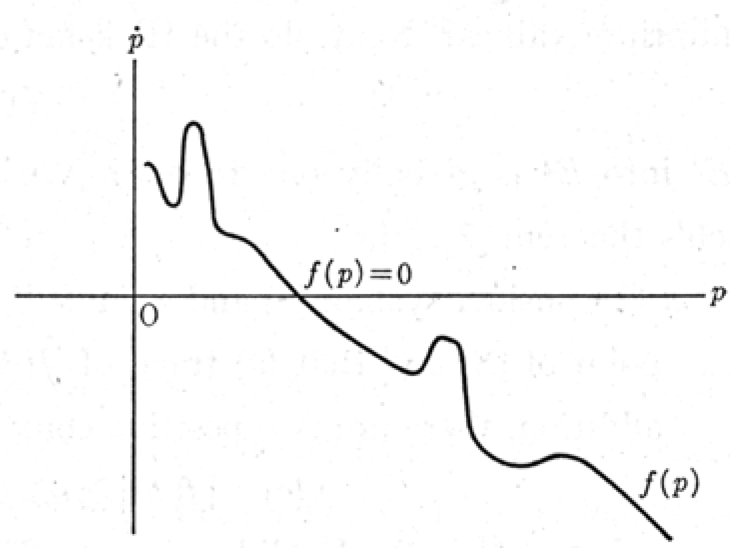


Figure 2



We are interested, in what follows, to know under what circumstances the conditions for global stability are satisfied as well as the conditions for global univalence being fulfilled. Although

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the analysis can be applied, in principle, to any economic transformation of a dynamic nature, we shall be particularly concerned with the competitive price adjustment mechanism.

I. Global Stability of Tâtonnement Price Adjustment with Three Commodities

Consider a standard tatonnement price adjustment system for three commodities. Suppose that the Walras law holds and that the excess demand function for each commodity is homogeneous of order zero. Then, if the third commodity is taken as a *numéraire,* the determination of equilibrium price ratios reduces to the study of the system of two differential equations,

$$\text{(S)}\quad \dot{p}=f(p),$$

Where $p=(p_1, p_2)$ are the deviations from their equilibrium prices and $\dot{p}=(\dot{p}_1, \dot{p}_2)$ are their time derivatives. $f(p)=(f_1(p_1, p_2), f_2(p_1, p_2))$ represents the excess demand functions and is of class C^1 on E^2.

In this system we are interested in two problems: First, what are the stability conditions in the large? Second, what are the conditions for global univalence? And how are they related?

Suppose $p=0=(0, 0)$ is an equilibrium point of (S) and assume the Jacobian matrix

$$J(p)=\left\{\frac{\partial f_i}{\partial p_k}\right\}$$

satisfies, at each point of E^2, the Hicksian conditions of perfect stability, i. e.,

$$\text{(i)}\quad \frac{\partial f_i}{\partial p_i}<0 \quad (i=1, 2)$$

and

$$\text{(ii)}\quad \det J(p)=\left(\frac{\partial f_1}{\partial p_1}\right)\left(\frac{\partial f_2}{\partial p_2}\right)-\left(\frac{\partial f_1}{\partial p_2}\right)\left(\frac{\partial f_2}{\partial p_1}\right)>0 \quad \text{on} \quad E^2.$$

Is then the solution $p=0$ of (S) asymptotically stable in the large or, in other words, does each solution curve of (S) approach 0 as $t\to\infty$? Does the actual price vector eventually approach the equilibrium values? Next, do the Hicksian conditions ((i) and (ii)) imply that the mapping of

$$\text{(T)}\quad y=f(p)$$

of E^2 into E^2 is globally one to one? We first present a lemma which is a revised version of Olech's theorem [7].

Lemma 1: Consider system (S) and assume that $f(p)$ is of class C^1 on E^2, that $p=0$ is an equilibrium point of (S) and that (a) trace of $J(p)$ is negative, while (b) det $J(p)$ is positive and, in addition, there are two positive constants ρ and γ such that

$$\text{(c)}\quad |f(p)|\geqq\rho>0 \quad \text{for} \quad |p|\geqq\gamma>0,$$

where $|p|$ is the Euclidean norm. Then the solution $p=0$ of (S) is asymptotically stable in the large.

Proof: See Appendix (1).

We also need,

Lemma 2: If (S) satisfies the conditions: (a) trace of $J(p)$ is negative, and (b) det $J(p)$ is positive and if the mapping (T) is globally one-to-one (global univalence), then the equlibrium solution $p=0$ is globally stable.

Proof: See Appendix (2).

We are now in a position to study whether or not the Hicksian conditions of perfect stability everywhere are sufficient for both global stability and global univalence. The answer is, in fact, affirmative. It should be noted, however, that the conditions (a), (b) and (c) in Lemma 1 do *not* guarantee the one-to-one-ness of (T), but only the fact that the equation $f(p)=y$, when y is from some neighborhood of (0, 0), has exactly one solution. The Hicksian conditions are more stringent conditions than (a), (b) and (c) in Lemma 1.

Theorem 1: If there exists an equilibrium point $f(0)=0$ and if the Hicksian conditions of perfect stability hold everywhere, then the equilibrium solution $p=0$ of (S) is globally stable and the mapping (T) is globally one-to-one.

Proof: See Appendix (3).

In fact, under the Hicksian conditions of perfect stability, the question of global stability is the same as the question of global univalence (one-to-one-ness in the large). In other words, if (S) is globally stable, then (T) is globally one-to-one and vice versa.

We are also interested to know whether or not the Hicksian conditions are necessary for global stability. We can show that even if the Hicksian conditions are not satisfied, the system can achieve global stability. Suppose that one of the Hicksian conditions is not satisfied, e. g., $\dfrac{\partial f_1}{\partial p_1}\equiv 0$ and none of the other coefficients of excess demand functions vanishes, then we can still show that the system is globally one-to-one and globally stable, as long as (a) and (b) are satisfied. From the assumption we have $\left(\dfrac{\partial f_1}{\partial p_2}\right)\left(\dfrac{\partial f_2}{\partial p_1}\right)\neq 0$. Without loss of generality we can assume that $\left(\dfrac{\partial f_1}{\partial p_2}\right)\left(\dfrac{\partial f_2}{\partial p_1}\right)<0$ and that $\dfrac{\partial f_1}{\partial p_2}>0$ and $\dfrac{\partial f_2}{\partial p_1}<0$ on E^2, since the proof of the other possible case may be reduced to the above case simply by the change of variables. Using the proof of Theorem 1 (see the appendix), we have in this case,

$$g_+'(u)=\frac{\partial f_l}{\partial p_1} \quad \text{or} \quad g_+'(u)=\frac{\partial f_l}{\partial p_1}-\frac{\left(\dfrac{\partial f_l}{\partial p_2}\right)\left(\dfrac{\partial f_k}{\partial p_1}\right)}{\dfrac{\partial f_k}{\partial p_2}} \quad l=2,\ \ k=1.$$

And $g_+'(u)>0$ for $\bar{p}_1\leqq u<\bar{\bar{p}}_1$. Hence, in this case again $\phi(\bar{p}_1)\neq\phi(\bar{\bar{p}}_1)$. This means, again, that $f_2(\bar{p})\neq f_2(\bar{\bar{p}})$, which contradicts $f(\bar{p})=f(\bar{\bar{p}})=b$. So (S) is globally stable. In summary, we have,

Theorem 2: The Hicksian conditions of perfect stability are *not* necessary for global stability—— also for global univalence. As long as the sum of the own effects is negative and det $J(p)$ is positive everywhere, while any one of the four effects (two own effects and two cross effects) is negligible, then (S) is globally stable and (T) is globally one-to-one[1].

1) Compare this result with that of the system which contains gross complementary goods in Sato [8].

Thus far the results are presented in terms of the Jacobian matrix (thus, the Hicksian conditions). The next theorem deals with stability analysis independent of the Jacobian matrix.

Theorem 3: Suppose that $f(p)$ in (S) is continuous and the coefficients of the excess demand functions $\frac{\partial f_i}{\partial p_i}$ $(i=1, 2)$ exist and are continuous on E^2. Assume further that: (1) (S) has exactly one equilibrium point $p=0$ and it is a point of attraction; (2) there are two positive constants ρ and γ such that

$$|f(p)|\geqq\rho \quad \text{for} \quad |p|\geqq\gamma,$$

and

$$(3) \quad \sum_{i=1}^{2}\frac{\partial f_i}{\partial p_i}\leqq 0 \quad \text{on} \quad E^2.$$

Then $p=0$ is globally stable.

Proof: It is enough to note that the existence and continuity of $\frac{\partial f_i}{\partial p_i}$ $(i=1, 2)$ suffice for Green's formula to hold. Assumption (1) gives us the fact that the boundary of the set of attraction Ω does not contain any singualr point of S.

II Multi-Commodity Case

We can extend the analysis to a more general case of $n+1$ commodity case. Consider now a standard tâtonnement price adjustment system for n commodities (see Sato [8]):

$$(\mathcal{S}) \quad \dot{p}=f(p).$$

Theorem 4: Let $f(p)$ be an n-dimensional excess demand vector function of class C^1 on E^n such that

$$f(0)=0; \; f(p)\neq 0 \quad \text{if} \quad p\neq 0;$$

and that $p=0$ is a locally asymptotically stable solution of $(\mathcal{S})$. Assume further that

$$\text{(i)} \quad \alpha(p)\leqq 0, \quad \alpha(p)=\max(\lambda_i(p)+\lambda_j(p)), \; 1\leqq i<j\leqq n,$$

where $\lambda(p)=(\lambda_1(p), \lambda_2(p), \cdots, \lambda_n(p))$ are the characteristic roots of the symmetric part of the Jacobian $J(p)$, i. e., $0=|\lambda I-H(p)|$, $H(p)=(J+J^*)/2$, and

$$\text{(ii)} \quad \int^{\infty}[\min_{|p|=\rho}|f(p)|]d\rho=\infty.$$

Then, $p=0$ is a globally asymptotically stable solution of $(\mathcal{S})$.

Proof: See Appendix (4).

A more traditional result will be obtainted by studying the Jacobian of $f(p)$. That is to say,

Theorem 5: Let $f(p)$ be of class C^1 on E^n and let $J(p)$ be the Jacobian matrix of f. Let $H(p)$ be a Hicksian matrix for all $p\neq 0$, where $p=0$ is a stationary point, $f(0)=0$. Then every solution of $(\mathcal{S})$ is globally stable.

Proof: Now $\lambda^n-TrH\lambda^{n-1}+\cdots+(-1)^n\det H=0$. Since the roots of H are all negative we have

$$|\lambda|<\beta_1, \;\; \beta_1>0 \quad \text{and also} \quad |\lambda_1\lambda_2\cdots\lambda_n|=|\det H|>\beta_2>0.$$

Thus each characteristic root everywhere satisfies $\lambda(p)<-\varepsilon$, for some constant $\varepsilon>0$ and meets the conditions set by Markus and Yamabe [5] and by Hartman [2]. For a global univalence condition we have

Theorem 6: Let a map T; $E^n \to E^n$ be given by $y=f(p)$. If $-\lambda(p)I-H$ is everywhere semi-positive definite, where λ is a positive, non-increasing function of γ for $\gamma>0$ such that

$$\int^{\infty} \lambda(\gamma)d\gamma=\infty,$$

then T is globally one-to-one and onto.

Proof: If $J[s]=J(p_2 s+p_1(1-s))$, then

$$f(p_2)-f(p_1)=(\int_0^1 J[s]\mathrm{ds})(p_2-p_1).$$

Hence, for any constant, symmetric, positive definite matrix G,

$$(p_2-p_1)G(f(p_2)-f(p_1))=\int_0^1 (p_2-p_1)GJ[s](p_2-p_1)ds.$$

For example, if GJ is negative definite and $p_1 \neq p_2$ then the integral is negative so that the map T; $E^n \to E^n$ given by $y=f(p)$ is one-to-one and onto.

For a special case of the Jacobian in which the coefficients of the excess demand functions satisfy the quasi-dominant-diagonal conditions, we obtain more useful theorem.

Theorem 7: If the coefficients of the excess demand functions are quasi-dominant-diagonal for all $p \neq 0$, then every solution of ($\mathcal{S}$) is globally stable and (T) is globally one-to-one.

Proof: Since J satisfies the quasi-diagonal conditions everywhere, (T) is a contraction mapping and ($\mathcal{S}$) exhibits a globally stable solution and (T) itself is one-to-one (in view of the Gale-Nikaido theorem [1]).

The above is one of the most powerful results in which the global Hicksian conditions alone guarantee global univalence as well as global stability.

Samuelson's condition for the Jacobian being quasi-definite can be extended to a general nonlinear adjustment system in order to obtain the conditions for both global stability and global univalence. If the Jacobian is quasi-definite everywhere so that the symmetric part is negative definite, it satisfies the Krasovski theorem for global stability and also the mapping is globally one-to-one. In view of the theorem due to Samuelson [9, p. 141], if the Jacobian is quasi-definite, it is necessarily Hicksian. Hence, we have,

Theorem 8: If the coefficients of the excess demand functions are quasi-definite for all $p \neq 0$, every solution of ($\mathcal{S}$) is globally stable and the mapping is globally one-to-one. The matrix of excess demand coefficients is everywhere Hicksian.

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APPENDIX

(1) Proof of Lemma 1:

Proof: We denote by $p(t, Q)$ the solution of (S) passing through Q; that is $p(0, Q)=Q$, where Q

$\in E^2$, $\dot{p}(t, Q)=f(p(t, Q))$. Let Ω denote the set of attraction of the equilibrium point (0, 0) of (S); that is, Ω is the set composed of all solution curves of (S) which approach (0, 0) as $t \to \infty$. By (a) and (b), (0, 0) is an equilibrium point of attraction of (S), and so Ω contains some neighborhood of (0, 0) and consequently, Ω is a non-empty open set. To prove the lemma, we have to show that $\Omega = E^2$. Suppose that $\Omega \neq E^2$. Then bd $\Omega \neq 0$. Since Ω is composed by solution curves of (S) the same holds for the boundary bd Ω of Ω. Let $Q \in$ bd Ω. Then $p=p(t, Q) \in$ bd Ω for $0 \leqq t < \omega(Q)$. Since each equilibrium point of (S) is a point of attraction, bd Ω cannot contain any equilibrium point of (S) and, consequently, $p(t, Q)$ cannot approach any equilibrium point of (S). Now by (c), the p-set of equilibrium points is compact. This, together with (c), shows the existence of $0<d<\rho$ and $\eta>0$ such that $|f(p)| \geqq d$, which contradicts the fact that $Q \in$ bd Ω. Therefore, bd Ω is empty and Lemma 1 is proved.

(2) Proof of Lemma 2:

Proof: Suppose there exists a vector function $h(p)$ such that it is of Class C^1 on E^2 for which inequalities (a) and (b) hold on E^2, and such that (T) is not globally one-to-one on E^2. That means there are $\bar{p}$ and $\bar{\bar{p}}$, $\bar{p} \neq \bar{\bar{p}}$ and

$$h(\bar{p}) = h(\bar{\bar{p}}) = a.$$

Then the vector functoin, $f(p)=h(p)-a$, would satisfy all assumptions of the global stability problem and at the same time system (S) would have two equilibrium points. It is obvious that no equilibrium point of (S) (if there are two or more) can be globally stable. Thus the counter example to the global stability problem, if there exists one, will serve also as a counterexample to the global univalence problem. On the other hand, if $f(p)$ satisfies (a) and (b), $f(0)=0$, and the mapping (T) is globally one-to-one, then (c) holds and, by Lemma 1, $p=0$ is globally stable. In other words, if the global stability conditions are satisfied, then, because of Lemma 1, the global univalence conditions are satisfied.

(3) Proof of Theorem 1:

Proof: From the assumption we have $\left(\frac{\partial f_1}{\partial p_1}\right)\left(\frac{\partial f_2}{\partial p_2}\right)>0$. Suppose that (T) is not globally one-to-one. Then there are two points $\bar{p} \neq \bar{\bar{p}}$ such that $f(\bar{p})=f(\bar{\bar{p}})=\mathrm{b}$. For simplicity, assume that $b=0$. From the first part of the Hicksian conditions, $\frac{\partial f_1}{\partial p_1}<0$ and $\frac{\partial f_2}{\partial p_2}<0$, one can conclude that if $\bar{p}=(\bar{p}_1, \bar{p}_2)$ and $\bar{\bar{p}}=(\bar{\bar{p}}_1, \bar{\bar{p}}_2)$, then $\bar{p}_1 \neq \bar{\bar{p}}_1$ and $\bar{p}_2 \neq \bar{\bar{p}}_2$. Assume further that $\bar{p}_1 < \bar{\bar{p}}_1$ and $\bar{p}_2 < \bar{\bar{p}}_2$. Consider the rectangular region

$$\pi = \{p;\ \bar{p}_i < p_i < \bar{\bar{p}}_i,\ i=1, 2\}.$$

Let function $\phi(u)$ be defined for $\bar{p}_1 \leqq u \leqq \bar{\bar{p}}_1$ as follows:

$$\phi(u)=\begin{cases} \bar{p}_2 & \text{if } \ f_2(u, p_2)<0 \quad \text{for} \quad \bar{p}_2<p_2<\bar{\bar{p}}_2, \\ v & \text{if there is } \ \bar{p}_2<v<\bar{\bar{p}}_2 \quad \text{that} \quad f_2(u, v)=0, \\ \bar{\bar{p}}_2 & \text{if } \ f_2(u, p_2)>0 \quad \text{for} \quad \bar{p}_2<p_2<\bar{\bar{p}}_2. \end{cases}$$

Because of the Hicksian conditions, in particular $\frac{\partial f_2}{\partial p_2}<0$, it is easy to see that $\phi(u)$ is well defined and because of $f(\bar{p})=f(\bar{\bar{p}})=b$, we have

$$\phi(\bar{p}_1)=\bar{p}_2, \quad \phi(\bar{\bar{p}}_1)=\bar{\bar{p}}_2.$$

It is easy to see also that $\phi(u)$ is continuous on $[\bar{p}_1, \bar{p}_2]$. We now prove that the right-hand derivative of $\phi(u)$ exists for $\bar{p}_1 \leqq u < \bar{\bar{p}}_1$. Indeed, if $\bar{p}_2 < \phi(u) < \bar{\bar{p}}_2$ for some u_o, then even the derviative $\phi'(u)$ exists, since in that case $f(u, \phi(u)) \equiv 0$ in some neighborhood of u_o. We have then

$$\phi'(u_o)=-\frac{\partial f_2}{\partial p_1}\Big/\frac{\partial f_2}{\partial p_2}\bigg|_{p_1=u_o,\, p_2=\phi(u_o)}$$

If $\phi(u_o)=\bar{p}_2$ (or $\bar{\bar{p}}_2$) and $f_2(u_o, \phi(u_o)) \neq 0$, then $\phi(u)$ is constant in some neighborhood of u_o and, therefore, the derivative of $\phi(u)$ also exists and is zero. Finally, consider the case when $\phi(u_o)=\bar{p}_2$ (or $\bar{\bar{p}}_2$) and $f_2(u_o, \phi(u_o))=0$. Then we have $\phi(u)=\max(\bar{p}_2, q(u))$, (or $\min(\bar{\bar{p}}_2, q(u))$, for $u_o \leqq u < u_o+\varepsilon'$ where by $q(u)$ we denote the function satisfying the conditions $q(u_o)=\phi(u_o)$ and $f_2(u, q(u)) \equiv 0$. It follows that $\phi_+'(u_o)$ exists and is equal to $q(u_o)=0$. Therefore, $\phi_+'(u)=0$ or $\phi_+'(u)=-\frac{\partial f_2}{\partial p_1}\Big/\frac{\partial f_2}{\partial p_2}$, in the formula $p_1=u$, $p_2=\phi(u)$. Consider now the function $g(u)=f_1(u, \phi(u))$. Since $\phi_+'(u)$ exists and f_1 is of class C^1, $g_+'(u)$ also exists and is equal to $g_+'(u)=\frac{\partial f_1}{\partial p_1}$ or $g_+'(u)=\frac{\partial f_1}{\partial p_1}-\left(\frac{\partial f_1}{\partial p_2}\right)\left(\frac{\partial f_2}{\partial p_1}\right)\Big/\frac{\partial f_2}{\partial p_2}$. From the assumption we have $g_+'(u)<0$ for $\bar{p}_1 \leqq u < \bar{\bar{p}}_2$. Hence, $g(\bar{p}_1) \neq g(\bar{\bar{p}})$. This means that, owing to $\phi(\bar{p}_1)=\bar{p}_2$ and $\phi(\bar{\bar{p}}_1)=\bar{\bar{p}}_2$, $f_1(\bar{p}) \neq f_1(\bar{\bar{p}})$, which contradicts with the fact that $f(\bar{p})=f(\bar{\bar{p}})=b$. Thus (T) is globally one-to-one and (S) is globally stable.

(4) Proof of Theorem 4:

Proof: It suffices to show that if (i) and (ii) hold, $f(p)$ satisfies a theorem due to Hartman and Olech [3]. For fixed p and a pair of constant vectors u and w, we have

$$\begin{aligned}&(Jw \cdot w)|v|^2+|w|^2(Jv \cdot v)-(v \cdot w)[(Jv \cdot w)+(v \cdot Jw)]\\ &\quad \leqq \alpha[|v|^2|w|^2-(v \cdot w)^2],\\ &v=f(p), \quad |w|=1 \quad \text{and} \quad w \cdot f(p)=0.\end{aligned}$$

The left-hand side of the above is unchanged if J is replaced by its symmetric part H, and v, w are subjected to an orthogonal transformation: so that, without loss of generality, it can be supposed that $H=\operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ at the given point p. The left-hand side is then seen to be

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i(w^iw^iv^jv^j+w^jw^jv^iv^i-2v^jw^jv^iw^i)\\ &=\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i(w^iv^j-w^jv^i)^2,\end{aligned}$$

which is

$$\frac{1}{2}\sum_{i \neq j}\sum(\lambda_i+\lambda_j)(w^iv^j-w^jv^i)^2$$

$$\leqq \frac{1}{2}\alpha \sum_{i=1}^{n} \sum_{j=1}^{n} (w^i v^j - w^j v^i)^2.$$

The above will satisfy the conditions set by Hartman and Olech for global stability.

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