A COMPARISON OF THE PROPERTIES OF ESTIMATED NON-LINEAR AND LINEARIZED MACROECONOMIC MODELS

ASATOSHI MAESHIRO AND JOHN J. SPITZER[1]

I. *INTRODUCTION*

The purpose of this paper is to report the preliminary result of maximum likelihood estimation of two econometric models, one containing a nonlinear behavioral equation and the other containing a linearized version of the nonlinear equation of the first model, and to compare the forecasting and dynamic properties of the two models.

Since the seminal work of Tinbergen[2], many econometric models of a national economy have been constructed, but in most models, structural equations are expressed as linear in coefficients even if the intrinsic economic nature demands the functional form be nonlinear in the coefficients. The main reason for this is the computational burden, and the main justification for this, if there is a justification at all, is the loosely interpreted Taylor's series expansion theorem. In addition, the linearized structural equations do quite often fit well to the data. The same has been true in numerous single-equation models utilized in quantitative economic research in all fields of economics.

Now that a computer program of nonlinear full-information maximum likelihood (FIML) method exists[3] and is available for the estimation of not only a single nonlinear structural equation but also a system of nonlinear structural equations, at least the excuse for linearization due to computational burden cannot be accepted. It is also possible that the implication derived from an estimated linearized model is different from the one derived from the estimated original nonlinear model. The high squared multiple correlation (R^2) obtained for an individual structural equation in a linearized model may not necessarily be an indication of good approximation to the original nonlinear model.

In studying the effect of linearization of one nonlinear structural equation in the original model, one *may* proceed by estimating the remaining linear equations by a single equation method such as the two-stage least squares method, in which case the two estimated models will be identical except for the particular structural equation under scrutiny. This procedure, however, is

---

1) The authors are respectively associate professor of economics and graduate student in economics at the University of Pittsburgh. Computations for the paper were done at the Computer Center of the University of Pittsburgh. Computer time was supplied by the Computer Center under a grant from the National Science Foundation.

2) Tinbergen, J., *Statistical Testing of Business Cycle Theories,* Geneva: League of Nations, 1939.

3) Bard, Y., *Non Linear Parameter Estimation and Programming,* Contributed Program Library #360D 13.6.003. IBM Corp. 1967.

not followed. Instead, the whole system is estimated simultaneously using the maximum likelihood method and hence, how the difference in one structural equation——nonlinear versus linearized——affects the estimated result of the remaining section of the model will become an interesting question.

In a different context, in the leading article of a recent issue of *Econometrica*, Klein[4] compared alternative estimation techniques as applied to the Klein-Goldberger model[5] and concluded that "We *can* make FIML estimates of fairly large sophisticated systems, and we *can* deal with nonlinear structures. From a technological point of view, I regard these as definite forward steps in econometric development..." (p. 182). Our study may be considered as a further implementation of his point.

II. *MODELS AND ESTIMATED RESULTS*

The model used in this study (see Appendix) is a modified and extended version of Klein's Model I[6]. Klein's Model I has been selected as the basis of the model used because (1) it is the model that has been used most frequently in expository studies of estimation and application[7], (2) its properties are well known, and (3) it is of manageable size.

There are two differences between the model used in this study and Klein's Model I. First, our model uses different explanatory variables in the consumption function, i. e., personal disposable income and lagged consumption are used in place of current and lagged non-labor income, and the aggregate wage bill. This change was made mainly because compatible data for the post-war periods could not be compiled and unreasonable estimates for the original equation were obtained when proxy variables were used. Second, our model contains three structural equations in addition to the three structural equations of Klein's model. An imoprt function, a production function, and a cost-minimizing behavioral constraint on the production function have been added.[8] Necessary definitional and accounting identities have, of course, been added, such that our model has a total of fourteen equations.

4) Klein, L. R., "Estimation of Interdependent Systems in Macroeconometrics," *Econometrica*, April, 1969, pp. 171–192.

5) Klein, L. R. and A. S. Goldberger, *An Econometric Model of the United States*, 1929–1952, Amsterdam: North Holland, 1955.

6) Klein, L. R., *Economic Fluctuations in the United States*, New York: John Wiley & Sons, 1950.

7) See for example: Theil, H., and J. C. G. Boot, "The Final Form of Econometric Equation Systems," *Review of International Statistical Institute*, Vol. 30, 1962, pp. 136–152; van den Bogard, P. J. M. and H. Theil, "Macrodynamic Policy Making," *Metroeconomica*, Vol. 11, 1959, pp. 236–302, both of which are reprinted in *Readings in Economic Statistics and Econometrics*, (edited by A. Zellner) Boston: Little, Brown and Company, 1968; Herman, C., & N. Divinsky, "The Computation of Maximum-Likelihood Estimation" in *Studies in Econometric Method*, Cowles Commission Monograph 14 (edited by W. C. Hood and T. C. Koopmans) New York: John Wiley and Sons, 1953, pp. 236–302.

8) The addition of the production function and the cost-minimizing behavioral constraint necessitates reinterpretation of the wage equation in the original model, which was derived by maximizing the *anticipated* profit and is interpreted as the demand equation for labor. One way is to interpret the two new equations as a realization (or modification) function to the original wage equation, which is a well-known concept in the study of the investment function. Another interpretation is to take the original wage equation as a

The form of the production function is the focal point of this study. Although a good case may be made for nonlinearity in the coefficients for other structural equations in the model, production functions such as the Cobb-Douglas[9] and CES[10] production function have specifically been derived from sound economic premises, such that these functions are intrinsically nonlinear in the parameters. Although the Cobb-Douglas function can be transformed into a log-linear form for the purposes of estimation (provided the disturbance term in the original form is multiplicative), there is no reason why this must be done.

In the present study, two forms of our model will be tested. One form will have a linearized version of the production function along the lines of the production function in the Klein-Goldberger model,[11] and will be called the "naive" model.[12] The second model will contain a CES production function with a multiplicative disturbance term. This form of the CES production function was chosen partly because it represents a more complicated family of nonlinear production functions than does the Cobb-Douglas function, and partly because its forecasting performance, based on another study,[13] (in which the alternative forms of the nonlinear production function were compared in the context of the simultaneous equation model), was revealed to be superior.

Before presenting the estimated results, a mention should be made of the cost minimizing constraint which contains the parameters of the production function and hence makes the estimation more restrictive and more nonlinear. The rationale of attaching this constraint is found in the studies of Bodkin and Klein[14] and Diwan.[15] Diwan argues that the production function "... is simply a technical relationship and does not depend on economic factors such as prices. As a result the relation does not necessarily limit the isoquant to the positive and hence the economic quadrant of the factor space." (p. 443) Diwan stresses that the constraint equation is a "behavioristic relation." (p. 444) The addition of the constraint, however, raises one pro-

historically or institutionally determined (e. g., through the labor-union bargaining) relationship existing aside from the profit-maximization of the firm since no parameter in the wage equation is *explicitly restricted* by the profit-maximization procedure. Then the two new equations may be interpreted as a representation of the force exerted by the firm trying to minimize the cost.

9) Cobb, C. W. & P. H. Douglas, "A Theory of Production," *American Economic Review*, Supplement, March, 1928, pp. 139–165.

10) Arrow, K. J., et. al., "Capital-Labor Substitution and Economic Efficiency," *The Review of Economics and Statistics*, August, 1961, pp. 225–250.

11) Klein, L. R. and A. S. Goldberger, *op. cit.* p. 5.

12) Note that while each structural equation in the model is linear, the reduced form of the model is not linear——the impact multipliers contain variables as well as parameters due to nonlinear definitional relationship.

13) Spitzer, J. J., *Forecasting Efficiency and Nonlinear Aggregate Economic Models*, Unpublished Ph. D. Thesis, University of Pittsburgh, 1971.

14) Bodkin, R., and L. R. Klein, "Nonlinear Estimation of Aggregate Production Functions," *Review of Economics and Statistics*, February, 1967, pp. 28–44.

15) Diwan, R., "Alternative Specifications of Economics of Scale," *Economica*, November, 1966, pp. 442–453.

blem.[16] (Also see footnote 8.) In order to make the system complete, one needs an additional endogenous variable, the selection of which is somewhat arbitrary, and which influences the result of estimation.[17] After several experiments, indirect business taxes have been made endogenous for the purpose of this study.

Table 1 contains the results of the estimation of the five structural equations. (The constraint equation has been omitted since the parameters in it are the same as in the production function.) The parameter estimates were obtained by applying the nonlinear method of FIML to both models. The data used are annual data covering the periods 1929–1941 and 1946–1966. For the model with the linear production function, two versions have been estimated, one with the cost-minimizing constraint (headed by the term "Constrained Naive" in the table) and the other with no such constraint (headed by the term "Nonconstrained Naive"). The main reason for examining two versions of the model with the linear production function is that the cost-minimizing constraint places an undue restriction on this model; i. e., the constraint requires the constancy of the ratio of the wage rate to the rental rate of capital. The historical record does not necessarily reveal this constancy, and hence the model with this constraint may be unduly punished. Interpreted from a different perspective, it may be that a person who is willing to use a linear production function, may not go so far as to accept the implication of this constraint. Therefore, one version is estimated without this constraint, although this deprives us of strict comparability with the model containing the CES function and its corresponding constraint.[18]

Examination of Table 1 reveals that the difference in the form of the production function does affect the overall estimated results of the *other* equations noticeably but not drastically. One interesting observation is that the overall estimated results of the model containing the constrained linear production function are more similar to the estimates of the model with the CES production function than to those of the model containing the nonconstrained linear production function.

The squared multiple correlation coefficient, R^2, obtained for each equation is very high without exception in all three models. Taken by itself, each coefficient estimate of any of the three models has the *a priori* expected sign and its magnitude is not unreasonable. For example, the respective short-run marginal propensities to consume are of the magnitude of .414, .534, and .619, while the corresponding equilibrium (or stationary) marginal propensities to consume are .971. .940, and .924. Each of these estimates are well within the realm of the estimates obtained

16) Note that the constraint equation contains no new endogenous variables. Hence the system becomes overdetermined; there is a shortage of endogenous variables.

17) Bodkin, R. and L. R. Klein, op. cit., p.13. The authors indicate that in their two equation study, the choice of endogenous variables greatly changes the magnitude of certain parameters. A change in exogenous-endogenous relationships changes the direction of minimization, which in turn changes the parameter estimates.

18) Because the unconstrained naive model does not create the problem mentioned above in footnote 16, the endogenous variables in this model are different than in the other two models. The arbitrary variable, indirect business taxes, as well as the wage rate and the rental rate of capital have been omitted. Hence, strict comparability is limited. See also footnote 17.

Table 1

Estimates of Non-linear and Linearized Models

Consumption:	C	$=A$	A_{10}	+	$A_{11}Y_d$	+	$A_{12}C_{-1}$			
(CES model)			.468		.414		.574			$R^2=.9988$
t-ratio			(.28)		(8.85)		(10.34)			
(Constrained Naive)			1.271		.534		.432			$R^2=.9990$
t-ratio			(.91)		(14.45)		(9.87)			
(Non-constrained Naive)			2.625		.619		.330			$R_2=.9990$
t-ratio			(1.921)		(14.94)		(6.70)			
Investment:	$(I\text{-}D)$	=	A_{20}	+	$A_{21}(\pi+P)$	+	$A_{22}(\pi+P)_{-1}$	+	$A_{23}K_{-1}$	
(CES Model)			14.203		.355		.513		−.066	$R^2=.9044$
t-ratio			(3.17)		(4.76)		(6.46)		(11.17)	
(Constrained Naive)			13.302		.555		.279		−.063	$R^2=.9352$
t-ratio			(3.48)		(7.741)		(3.602)		(10.66)	
(Non-constrained Naive)			17.545		.440		.458		−.072	$R^2=.9156$
t-ratio			(3.84)		(5.39)		(5.17)		(10.10)	
Imports:	M	=	A_{30}	+	$A_{31}Y_d$	+	$A_{31}M_{-1}$			
(CES Model)			−1.645		.018		.854			$R^2=.9747$
t-ratio			(2.99)		(2.67)		(8.82)			
(Constrained Naive)			−1.978		.032		.648			$R^2=.9749$
t-ratio			(3.56)		(5.58)		(8.33)			
(Non-constrained Naive)			−2.051		.038		.544			$R^2=.9736$
t-ratio			(3.57)		(6.11)		(6.25)			
Private Wages:	W_1	=	A_{40}	+	$A_{41}(Q-W_2)$	+	$A_{42}(Q-W_2)_{-1}$	+	$A_{43}(t-1928)$	
(CES Model)			−10.129		.398		.124		.448	$R^2=.9992$
t-ratio			(8.50)		(24.83)		(8.03)		(4.52)	
(Constrained Naive)			−9.378		.297		.214		.623	$R^2=.9988$
t-ratio			(6.61)		(14.15)		(10.08)		(5.02)	
(Non-constrained Naive)			−10.689		.352		.175		.410	$R^2=.9991$
t-ratio			(8.15)		(16.11)		(8.09)		(3.27)	
Private Product (CES): $(Q-W_2)=A_{50}e^{A_{51}(t-1928)}[A_{52}K*^{-A_{53}}+(1-A_{52})L^{-A_{53}}]^{-(A_{54}/A_{53})}$										
(CES Model)			$A_{50}=.051$		$A_{51}=0.19$		$A_{52}=.279$		$A_{53}=-.164$	$A_{54}=1.534$
t-ratio			(11.27)		(36.95)		(7.87)		(1.98)	(35.92)
										$R^2=.9962$
Private Product (linear):										
	$(Q-W_2)$	=	A'_{50}	+	$A'_{51}L$	+	$A'_{52}((K^*+K^*_{-1})/2)$	+	$A'_{53}(t\text{-}1928)$	
(Constrained Naive)			−331.258		2.522		.321		3.27	$R^2=.9771$
t-ratio			(20.09)		(27.68)		(27.65)		(8.72)	
(Non-constrained Naive)			−271.331		2.356		.243		4.78	$R^2=.9815$
t-ratio			(11.58)		(7.30)		(6.17)		(9.07)	

in other studies.[19)]

On the basis of the relative magnitudes of the coefficient estimates, and the high values of R^2, one may be tempted to hasten to the conclusion that even an intrinsically nonlinear function such as the production function can be well approximated by a linear form without drastically affecting the estimation results of the other structural equations, and that the extra effort and cost involved in the application of the nonlinear maximum likelihood method to a nonlinear econometric model is not worth-while. This temptation must be resisted until further analysis of such properties as multipliers, and dynamic properties have been undertaken, since the mild difference in the coefficient estimates along with the different forms of the production function may yield quite different results on these properties. These aspects will be examined in the next section.

19) For a comprehensive summary of empirical studies of the consumption function, see M. K. Evans, *Macroeconomic Activity*, New York: Harper & Row, 1969, Chapter 3.

III. *STABILITY, FORECASTING, AND IMPACT, DELAY AND CUMULATIVE MULTIPLIERS*

Stability

The inherent dynamic property of an estimated model can best be analyzed by examining the characteristic roots of the matrix of coefficients associated with the lagged endogenous variables in the reduced form.[20] Unless a model is linear in both coefficients and variables, the necessary matrix for this analysis can not be obtained directly, but must be linearly approximated by taking the total differential as is done by Goldberger.[21] Furthermore, in this case, the matrix varies at each time period depending on the values of the variables at which the derivatives are evaluated.

For this study, seven matrices for each model have been obtained using the values of the variables corresponding to the periods 1933, 1940, 1950, 1955, 1960, 1965, 1969, and their characteristic roots are computed. The largest real root (in absolute value) and the largest complex root (in absolute value) are given in Table 2.

Table 2

Stability and Characteristic Roots

Model	CES Model			Constrained Naive			Unconstrained Naive		
Year	Real	Complex		Real	Complex		Real	Complex	
		Modulus	Frequency*		Modulus	Frequency*		Modulus	Frequency*
1933	.866	.952	.171	1.044	no root		2.664	.260	.914
1940	.861	.930	.206	1.124	〃		〃	〃	〃
1950	.854	.901	.257	1.220	〃		〃	〃	〃
1955	.851	.872	.276	1.217	〃		〃	〃	〃
1960	.848	.834	.287	1.181	〃		〃	〃	〃
1965	.864	.818	.318	1 261	〃		〃	〃	〃
1969	.843	.763	.321	1.228	〃		〃	〃	〃

*Measured in radians

One immediately sees the instability of both estimated naive models. Thus the reasonable magnitude of each structural equation itself does not guarantee the stability of a model. Particularly the magnitude of the real root of the estimated unconstrained naive model is exceedingly high. In fact, further examination of the impact multipliers, delay multipliers, cumulative multipliers, and the final method forecasts (all of which will be defined later in the discussion) reveals such absurd results for this unconstrained naive model that it was excluded from further comparison. For example, the impact multiplier for government expenditure on GNP was computed as 7.44, and in the final method forecast the forecasted value of private GNP for the thirteenth period is thirty-thousand times as high as the actual value. On the other hand, although the estimated constrained naive model is also unstable, the magnitude of the largest root is close to unity and hence the degree of instability may not be of serious magnitude, at least for the short-run and intermediate analysis. Thus, further comparison of the performances of this model

20) See the article by Theil & Boot in reference in footnote 7 for an example.

21) Goldberger, A. S., *Impact Multipliers and Dynamic Properties of the Klein-Goldberger Model,* Amsterdam: North Holland, 1959, pp. 17–20.

and the model with the CES production function is pursued.

Referring once again to Table 2 and comparing the characteristic roots associated with the two constrained models, one observes, aside from the instability of the constrained naive model, an interesting fact that the constrained naive model has no complex root, i.e., no inherent cyclical property, while the model with the CES production function has a complex root as the dominant root. (There are small negative real roots in both models which give rise to some sawtooth movements.) This is a crucial observation, since apparently the linearization has changed one of the inherent properties of the original nonlinear model. In addition, further detailed study using the characteristic roots computed from the two models for all periods, thirty-seven in total, has revealed the following: with respect to the model with the CES production function, there is a very slight tendency for the largest real root to decrease over time and a strong tendency for the modulus of the largest complex root (in absolute value) to decrease; the largest real root associated with the naive constrained model tends to increase consistently over time, which foretells a more explosive nature of the performance of the model for the later periods.

Total Method Forecasts[22]

In total method forecasting, the forecast values of the current endogenous variables are obtained by inserting the actual values of the predetermined variables, i.e., the current exogenous, and lagged endogenous variables, into the estimated model and solving the resulting nonlinear simultaneous equation system for the values of the current endogenous variables. Since the equation system is nonlinear, an iterative procedure must be used.[23] Figures I and II show the computed results for selected variables covering the periods of 1929–1941 and 1946–1969 of which the last three forecast periods, 1967–1969, were not included in the sample used for estimation of the model. This dichotomous treatment of the periods was made in order to examine the predicting ability as well as the descriptive ability of a model.

From Figures I and II, it is observed that both models perform relatively well, especially for those periods included in the sample, but that for the periods 1967–1969, which are not included in the sample, the model with the nonlinear production function performs a little better. This is particularly true for investment, consumption, and private GNP. Another interesting point to observe is that the calculated values obtained from the "naive" model tend to consistently overestimate, while those from the CES model underestimate in the same period. In addition, the naive model tends to overestimate for several consecutive periods, and then underestimate for several consecutive periods. To supplement the above analysis, the Theil's U-statistic[24] and the square root of the mean square error of the forecasts ($\sqrt{\text{MSEF}}$) are computed

22) Goldberger, A. S., *op. cit.*, pp. 49–51.

23) One method of solution of nonlinear simultaneous equations and the one which was used here is a Gauss-Newton iterative method. For an explanation, see McCracken, D. D. & W. S. Dorn, *Numerical Methods and FORTRAN Programming*, New York: John Wiley and Sons, 1964, p.p. 144–145.

24) A discussion and derivation of the Theil's U statistic can be found in Theil, H., *Economic Forecasts and Policy*, Amsterdam: North Holland, 1961, Chapter 2, Sections 4 and 5.

for all endogenous variables and are given in Table 3. The table reveals overall superior performance of the model with the CES production function.

FIGURE I

TOTAL METHOD FORECASTS MODEL CES——SELECTED VARIABLES

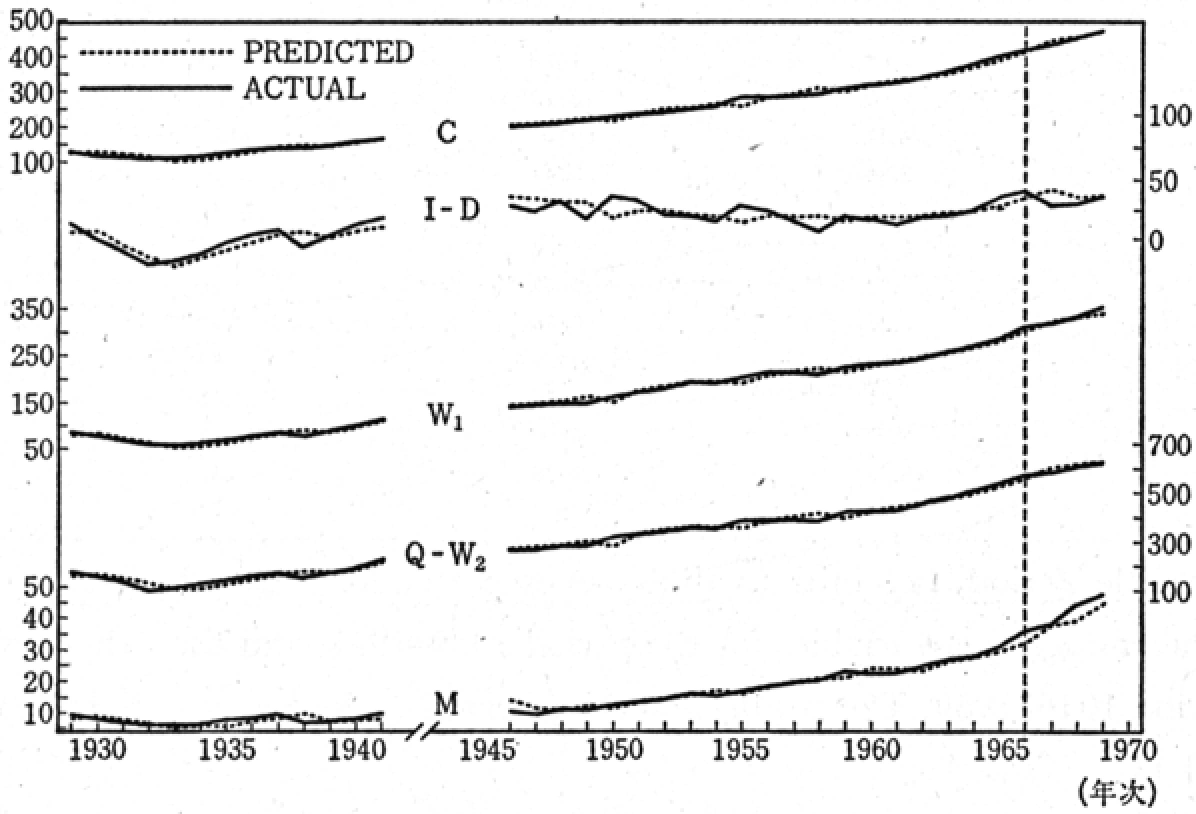
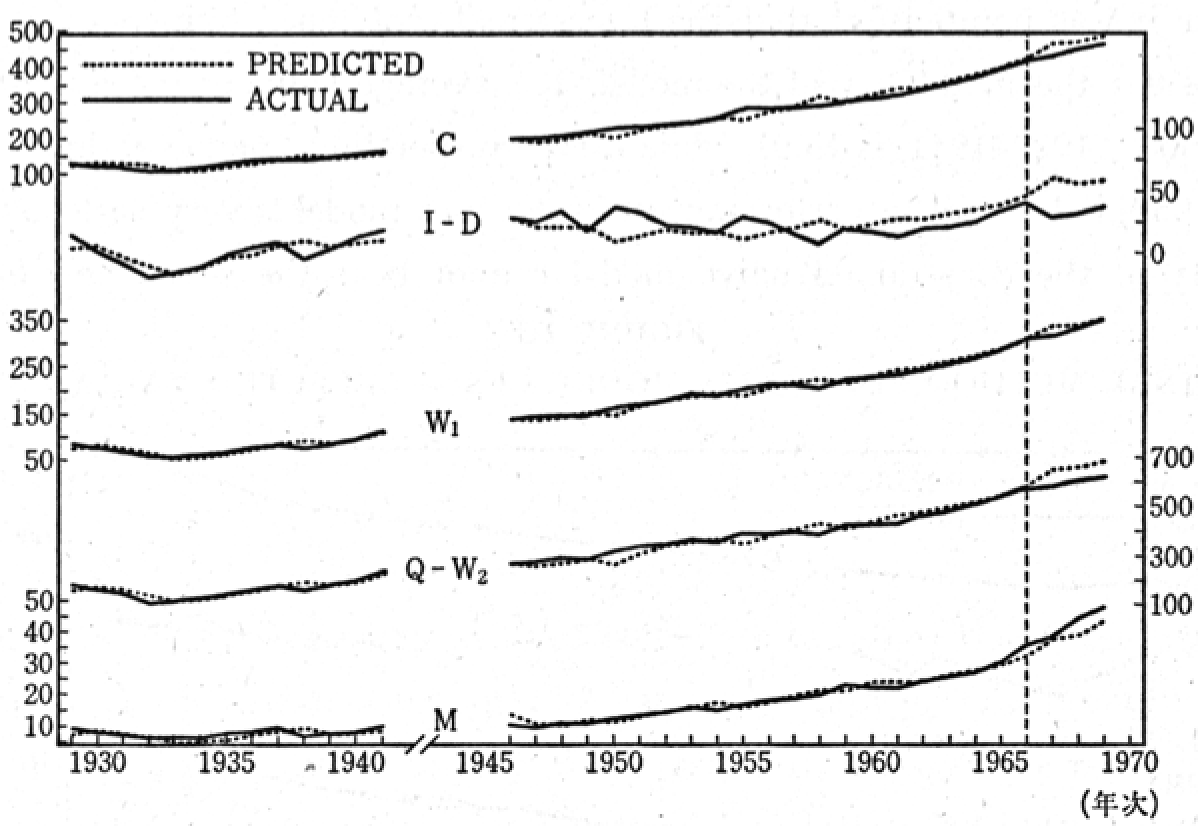


FIGURE II

TOTAL METHOD FORECASTS NAIVE MODEL——SELECTED VARIABLES



Final Method Forecast[25]

This method of forecasting provides a more severe test of the model, in particular its dynamic properties, since the values of the endogenous variables for all the periods except the initial one have to be generated by the model itself, using the initial value of the endogenous variables and the actual values of exogenous variables. In this particular study, due to the discon-

25) Goldberger, A. S., *op. cit.*, pp. 49–51.

Table 3

Total Method Forecast

Square Root of Mean Square Error of Forecast($\sqrt{\text{MSEF}}$) and Theil's U Statistic

Variable	(CES) $\sqrt{\text{MSEF}}$	(CES) U	(Constrained-naive) $\sqrt{\text{MSEF}}$	(Constrained-naive) U
C	8.28	.0092	32.43	.0349
I	7.68	.1018	26.16	.2799
W_1	7.60	.0113	12.96	.0190
$Q-W_2$	16.37	.0131	60.72	.0468
M	2.86	.0344	3.07	.0367
K	7.41	.0026	26.16	.0091
L	2.28	.0082	22.43	.0754
ρ	.02	.0081	.16	.0755
π	6.33	.0386	42.30	.2101
w	.02	.0020	.30	.0604
r	.01	.0176	.01	.0147
Y_d	9.92	.0100	53.71	.0514
K^*	26.71	.0096	252.16	.0843
T_i	9.23	.0677	7.13	.0530

tinuity of the sample period, two initial values are used, one for 1928 which is used to generate the values of the endogenous variables for the period 1929–1941, and the other for 1945 which is used for the period 1946–1969. The results for the selected endogenous variables are depicted in Figures III and IV.

From the figures one immediately observes an astonishing malperformance of the constrained naive model for the post-war period. This is what was foretold in the analysis of characteristic roots where it was pointed out that the largest real root tends to increase consistently over time and intensifies the instability of the model. The average value of the largest real roots for the pre-war periods, 1929–1941, is 1.091 while the corresponding average value for the post-war periods, 1946–1969, is 1.201. The performance of the other model is very satisfactory. Thus, the slight instability of the constrained naive model cannot bear the severe test of final method

FIGURE III

FINAL METHOD FORECASTS MODEL CES——SELECTED VARIABLES

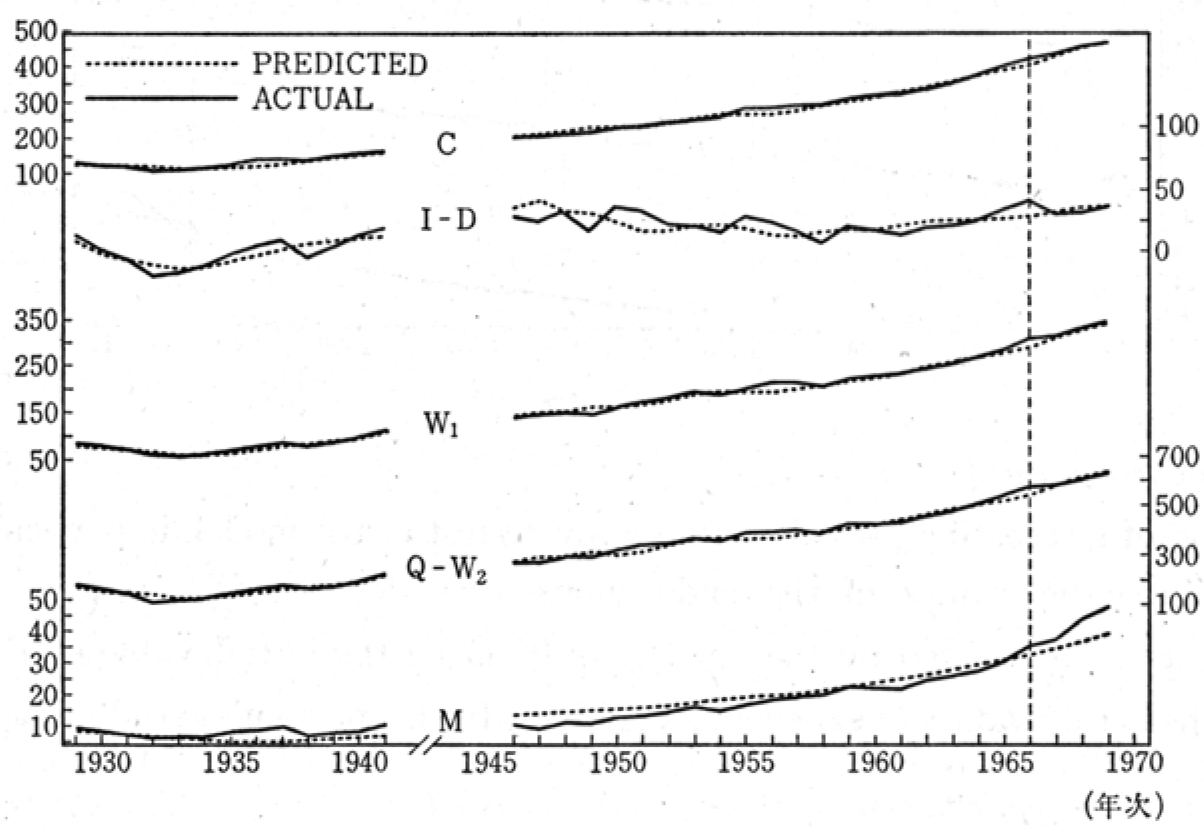


FIGURE IV

FINAL METHOD FORECASTS NAIVE MODEL——SELECTED VARIABLES

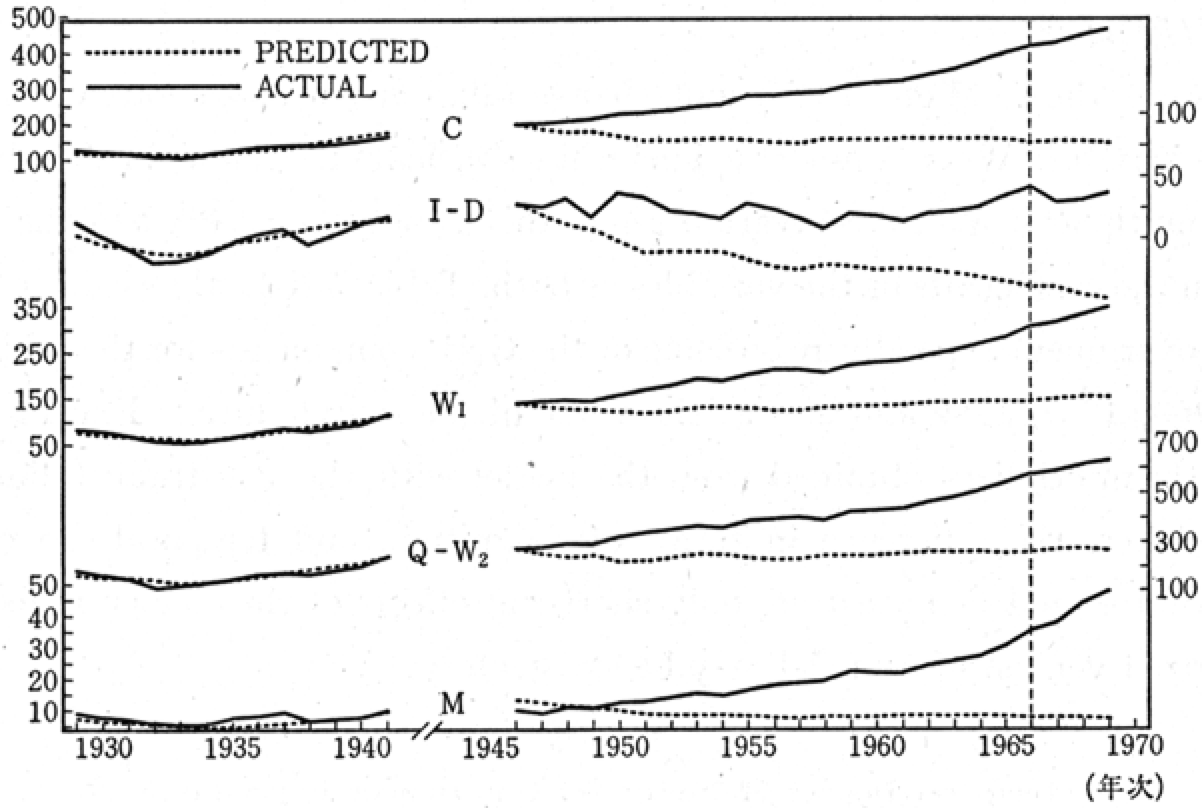


Table 4

Final Forecast Method

Square Root of Mean Square Error of Forecast and Theil's *U*-Statistic

Variable	(CES) $\sqrt{MSEF}$	U	(Constrained Naive) $\sqrt{MSEF}$	U
C	4.55	.0051	299.67	.4983
I	4.14	.0576	10.76	.9998
W_1	2.83	.0042	177.70	.3617
$Q-W_2$	12.25	.0098	340.43	.3798
M	6.27	.0766	35.26	.6661
K	4.72	.0017	933.28	.4881
L	1.25	.0045	7.03	.0261
ρ	.01	.0045	.05	.0261
π	6.25	.0383	7.91	.9992
w	.01	.0017	1.21	.2939
r	.01	.0180	.06	.2283
Y_d	4.25	.0043	338.74	.5178
$K*$	9.72	.0035	923.15	.5077
T_i	8.97	.0661	2.06	.0166

forecast. It might be argued, as many people have been arguing, that structural shifts must have taken place in the post-war economy, and hence the data for the pre-war and the post-war periods should not be combined directly in the estimation. This interpretation, at least in the context of this study, is misleading, since the model with the CES production function performs satisfactorily for the post-war periods as well; it may be evidence of the failure of linear approximation rather than structural shifts. Table 4 contains the square root of the mean square error of forecast ($\sqrt{MSEF}$), and Theil's U-statistics for all endogenous variables.

It may still be argued that for the practical application of an econometric model to policy formulation, what is important is to assess the correct value of impact (or short-run) multipliers and a few delay multipliers, and, that so long as these values are fairly well approximated by a linearized model, the linear approximation is still justifiable. In what follows, values of some multipliers obtained from the two models will be compared.

Impact Multipliers[26)]

This measures the total effect of a unit change in a current exogenous variable on a current endogenous variable. When a model is linear in coefficients as well as in variables, the magnitudes of the multipliers are time-invariant, but this is not the case when the model is non-linear, either in the coefficients or the variables or both. Table 5 lists the values of the impact multipliers of government expenditure on some of the GNP components for the periods 1933, 1940, 1950, 1955, 1960, 1965, 1969 and also at the value of the sample mean. Except for the general tendency that the multipliers obtained from the model with the constrained linear production function are higher, the difference in terms of magnitude and temporal movement is small. Thus, if one is interested in measuring only short-run effects of the change in exogenous variables, the linearized version of a model may be satisfactory.

Table 5

Temporal Impact Multipliers of Government Expenditure

Variable \ Year	1933	1940	1950	1955	1960	1965	1969	At the sample mean
GNP (Naive)	1.512	1.512	1.538	1.523	1.490	1.533	1.495	1.497
(CES)	1.492	1.480	1.483	1.465	1.435	1.456	1.421	1.481
Consumption (Naive)	.384	.384	.399	.391	.371	.397	.374	.375
(CES)	.387	.380	.382	.370	.351	.364	.343	.380
Investment (Naive)	.150	.150	.162	.156	.141	.160	.143	.144
(CES)	.122	·117	.118	.111	.099	.093	.093	.117

Delay Multipliers[27)]

Impact multipliers measure only the effect of a unit change in a *current* exogenous variable on *current* endogenous variables. Certainly, many economic policies have their effect throughout more than one period, and hence it is important to estimate properly the effect of current policies on the future course of the economy. Delay multipliers measure these effects. For example, a delay-3 multiplier measures the effect of a unit change occurring three periods ago, of an exogenous variable on a current endogenous variable.

Table 6 presents, for seven different terminal periods, 1933, 1940, 1950, 1955, 1960, 1965, and 1969, the value of the delay multiplier of government expenditure on GNP with delays of up to four periods, as well as the sum of the values of the four delays and the impact multipliers. The sum of these multipliers represents the cumulative effect of a sustained unit increase in government expenditure on the current GNP. For comparison, the impact multiplier,

26) Goldberger, A. S., *op. cit.*, pp. 14–20.

27) Goldberger, A. S., Econometric Theory, New York: John Wiley and Sons, 1964, pp. 373–378.

i. e. the delay-0 multiplier, is also listed.

Examination of the table reveals the following results. (1) The value of all the multipliers obtained from the constrained naive model is always greater than the corresponding value obtained from the other model, and the difference widens with the length of delay. (2) Without exception, the value of the delay multipliers obtained from the model with the CES production function decreases with the length of delay, while almost the opposite is true in the case of the constrained naive model.

Table 6

Delay Multipliers and Cumulative Multiplier of Government Expenditure on GNP

Terminal Year \ Delay		0	1	2	3	4	Sum
1933	(Naive)	1.512	.849	.783	.769	.854	4.767
	(CES)	1.492	.712	.612	.535	.494	3.845
1940	(Naive)	1.512	.889	.846	.799	.820	4.866
	(CES)	1.480	.713	.580	.449	.351	3.573
1950	(Naive)	1.538	.927	.926	.934	.927	5.252
	(CES)	1.483	.666	.500	.333	.198	3.180
1955	(Naive)	1.523	.924	.924	.940	1.049	5.360
	(CES)	1.465	.646	.456	.279	.140	2.986
1960	(Naive)	1.490	.894	.931	.972	1.068	5.355
	(CES)	1.435	.616	.422	.238	.099	2.810
1965	(Naive)	1.533	.974	1.037	1.110	1.217	5.871
	(CES)	1.456	.617	.394	.195	.057	2.719
1969	(Naive)	1.495	.921	.999	1.119	1.325	5.859
	(CES)	1.421	.563	.331	.123	.000	2.447

(3) Finally, the difference between the two corresponding values of the delay multipliers tends to widen with historical time, which is most clearly shown in the difference of the cumulative multipliers, e. g., .922 for 1933 while 3.412 for 1969. The last observation could be crucial since what is relevant for policy-formulation is presumably the relationship existing in the most recent periods.

IV. *CONCLUDING REMARKS*

On the basis of the models used and the results obtained in this study, the following may be concluded:

1. The linearization of a nonlinear structural equation may affect the overall estimate of a model noticeably when the nonlinear full information maximum likelihood method is used.

2. The linearization of a structural equation may change the inherent dynamic property of the estimated original model and may bring in an element of instability into the estimated new

model. In combination with the preceding remark, this suggests that when one obtains an unre asonable estimated linear model, before he accuses the estimation method, in particular the full information maximum likelihood method, or attributes the unreasonable result to data or structural shifts, one should also check the validity of the linear relationship imposed on some of the structural equations. For the particular model used in this study, the estimated model with the CES production function exhibits stability throughout the whole sample period but both models with the linear production function exhibit instability in their estimated results.

3. Regarding the forecasting performance of the actual models compared in this study, the one with the CES production function performs much better in the final method forecast.

4. Somewhat related to remark (2.), provided the true model is the one with a nonlinear structural equation, the model with a linearized structural equation may safely be used as an approximation in short-run analysis, e. g., impact multipliers, but it may not be approximated for intermediate or long-run analysis, e. g., delay multipliers and cumulative multipliers. Thus the linearized model could provide a misleading guide to, say, five-year economic planning.

Of course all the remarks made in the above are tentative and conditional. The performance of a nonlinear model and its linearized version, and the degree of approximation by the linearized model certainly depend on each individual case so that the generalization should not be made hastily from one example. What this investigation suggests, however, is that all those studies which utilized linear approximation, whether in econometric model construction or single equation model analysis, should be reexamined using the nonlinear full information maximum likelihood method.

(The University of Pittsburgh)

Appendix

The variables used will be listed below before a discussion of the model begins. The first 14 variables are the endogenous variables in the model, the remaining are exogenous.

Endogenous variables:

1. C = Consumption in billions of 1958 dollars
2. I = Gross private domestic investment in 1958 dollars
3. W_1 = Private wage bill in billions of 1958 dollars
4. $Q-W_2$ = Gross private product in billions of 1958 dollars
5. M. = Imports of goods and service in billions of 1958 dollars
6. K = Stock of capital goods available in billions of 1958 dollars
7. L = Billions of manhours employed
8. Y = Personal disposable income in billions of 1958 dollars
9. w^d = Hourly wage rate of private laborers
10. r = Rental rate of capital equipment (pure number)
11. K^* = Stock of capital currently employed in billions of 1958 dollars
12. π = Non-labor income excluding proprietors income in billions of 1958 dollars
13. ρ = Rate of employment of labor (1.-rate of unemployment)
14. T_i = Indirect business taxes in billions of 1958 dollars

Exogenous variables:

15. P = Proprietor's income in billions of 1958 dollars
16. t = Calendar year
17. N = Supply manhours in billions of manhours
18. D = Depreciation of capital equipment on replacement cost basis in billions of 1958 dollars
19. α = Proportion of proprietor's income which is non-labor income at time t
20. $G+X$ = Government expenditures and exports in billions of 1958 dollars
21. T_c = Corporate tax liability in billions 1958 dollars
22. IVA^c = Corporate inventory valuation adjustment in billions of 1958 dollars
23. IVA^p = Proprietor's inventory valuation adjustment in billions of 1958 dollars
24. Z_1 = Business transfer payments, statistical discrepancy, less subsides plus current surplus of government enterprises in billions of 1958 dollars
25. Z_2 = Contribution for social insurance+wage accruals less disbursements—business transfers+personal tax liability+corporate profits after corporate profits tax—transfers for government—interest paid by government—dividends.
26. D^* = Dummy variable., $D^* = \begin{cases} 1 & \text{Pre-world war II} \\ 0 & \text{Post world war II} \end{cases}$
27. U'_s = Disturbance terms

The Model

1. Consumption Function

$$C = A_{10}+A_{11}Y_d+A_{12}C_{-1}+U_c$$

2. Investment Function

$$I-D = A_{20}+A_{21}(\pi+P)+A_{22}(\pi+P)_{-1}+A_{23}K_{-1}+U_I$$

3. Imports Function

$$M = A_{30}+A_{31}Y_d+A_{32}M_{-1}+U_M$$

4. Wage Bill Function

$$W_1 = A_{40}+A_{41}(Q-W_2)+A_{42}(Q-W_2)_{-1}+A_{43}(t\text{-}1928)+U_W$$

5. Production Function

5-1 (non-linear)

$$(Q-W_2) = A_{50}e^{A_{51}(t-1928)}[A_{52}K^{*-A_{53}}+(1.0-A_{52})L^{-A_{53}}]^{-A_{54}/A_{53}}U_Q$$

5-2 (linear)

$$(Q-W_2) = A'_{50}+A'_{51}L+A'_{52}\left[\frac{K^*+K^*-1}{2}\right]+A'_{53}(t\text{-}1928)+U'_Q$$

6. Constraint Function:

$$\frac{r}{w}=\left(\frac{\partial(Q-W_2)}{\partial K^*}\bigg/\frac{\partial(Q-W_2)}{\partial L}\right)$$

6-1 (non-linear)

$$\frac{r}{w}=\left(\frac{A_{52}}{1.0-A_{52}}\right)\left(\frac{K^*}{L}\right)^{-(1.0+A_{53})}e^{A_{60}D^*}U_{r/w}$$

6-2 (linear)

$$\frac{r}{w}=\frac{A'_{52}}{2A'_{51}}+A'_{60}D^*+U_{r/w}$$

7. GNP

$$Q = C + I + G + X - M$$

8. National Income Identity

$$Q - T_i - D - IVA^P - IVA^C - Z_1 = \pi + P + W_1 + W_2 + T_C$$

9. Disposable Income

$$Y_d = \pi + P + W_1 + W_2 + IVA^P - Z_2$$

10. Manhours Available for Production

$$L = \rho N$$

11. Stock of Capital Used in Production

$$K^* = \rho K$$

12. Rental Rate of Capital

$$r = (\pi + D + T_C + \alpha P) / K^*$$

13. Wage Rate of Labor

$$w = (W_1 + (1 - \alpha) P) / L$$

14. Capital Stock Identity

$$K = K_{-1} + I - D$$