GROWTH AND STABILITY UNDER DIFFERENTIAL EFFICIENCY OF FACTOR INPUTS

RYUZO SATO*

It is well known that an economy consisting of *homogeneous* capital and labor inputs under the neoclassical conditions can in general achieve balanced growth with global stability (Solow [8]). However, the models consisting of the two-factor inputs cannot effectively explain how capital inputs attain higher productivity during the course of the investment period, nor do they take into consideration the fact that labor inputs can also achieve higher productivity by a round-about method of production, such as education, training and experience. The so-called "vintage" capital and "learning-by-doing" models [1] of economic growth are, to some extent, designed to deal with these problems, but they often fail to give any meaningful answers to the crucial question of productivity increase resulting from round-about methods of production, since the "vintage" and "learning-by-doing" effects enter only as exogenous factors.

In this paper we first examine the stability conditions of growth processes in an economy where there are a different age distributions of capital goods and a different efficiency distribution of labor inputs. This paper intends to study the property of growth equilibrium under the neo-classical conditions, when there are heterogeneous capital goods and heterogeneous labor inputs (such as skilled, semi-skilled and unskilled labor forces). It is an extension of the "vintage" and "learning-by-doing" models of economic growth in that we deal with the case of *endogenous* productivity increase of both factors. Unlike the *exogenous* models of heterogeneous labor inputs, however, we will explicitly introduce the "costs" of higher productivity of factor inputs due to round-about methods of production. Thus, the productivity increase (or decrease) of a factor input is internally, or endogenously, determined by the choice of the production method.

We begin with the case of two factor inputs whose supply conditions are endogenously determined, and then proceed to the case of multi-factor inputs whose supply conditions are also subject to internal choices of different techniques. The existence and stability conditions of global dynamic equilibrium will be derived for the general multi-factor input case. A set of sufficient conditions for stability is also presented, when the supply of one factor input is "limitational" in the sense that the supply conditions are exogenously determined. Various useful properties of the dynamic growth equilibrium under heterogeneous factor inputs will be investigated.

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I. *The Model*

We first illustrate the problems by using a simple model of neoclassical growth processes under the condition of endogenous (but homogeneous) labor input with a variable savings ratio. Thus, we have homogeneous but endogenous labor and capital inputs.

Production of income Y in an economy is carried by a constant returns to scale production function with homogeneous capital K and homogeneous labor L.

(1) $Y(t) = F[K(t), L(t)]$

where F is a continuous and differentiable (up to a desired order) function with $F_{K(t)}>0$, $F_{L(t)}>0$ and $F_{KK}<0$, $F_{LL}<0$ (positive marginal productivities with the law of diminishing marginal rate of substitution). The economic system is highly competitive and hence the factor inputs K and L are paid according to their marginal productivities. Accumulation of capital ΔK is made each year by saving a fraction (not necessarily constant over time) of national income

$\Delta K(t) = sY$

or (2) $K(t+1) = K(t)+sY(t), \quad 0 < s < 1.$

The fraction of saving out of income, s, varies according to whether the current profit rate (or the marginal productivity of capital) is greater than some long-run (or equilibrium) profit rate, i. e.,

(3) $s = s\{r(t)-\bar{r}\}$

where r=current profit rate$=\dfrac{\partial Y}{\partial K}$ and $\bar{r}$=the long-run profit rate$=\overline{\left(\dfrac{\partial Y}{\partial K}\right)}$. The function s is defined as an increasing function of $r(t)-\bar{r}$, i. e.,

$$\frac{ds}{d(r-\bar{r})} > 0$$

(4) $s = \bar{s}$ when $r = \bar{r}, \; 0 < \bar{s} < 1.$

The growth of labor is determined by

(5) $$L(t+1) = \left[1+n\left(\frac{Y}{L}\right)\right]L(t), \qquad n > 0$$

$$n' > 0$$

$$n'' \gtreqless 0.$$

To achieve any unambiguous results it is necessary that the population function be monotonic. Following Buttrick's analysis [2], it is assumed that $n'>0$. Thus, the population growth rate is an increasing function of per capita output. The second derivative, n'' will be unspecified although sometimes it plays an important role in the analysis of stability. The justification is that the economic effect on birth rate is of a smaller order so that overall the relationship between population growth rates and per capita income is monotonically increasing at all levels of the country's development (the effect on birth rates being only large enough to alter the sign of the second derivative——never enough to shift the sign of the first derivative of the population function).

The dynamics of this system is represented by the difference equations with two initial conditions $K(0)$ and $L(0)$;

(2) $K(t+1) = K(t) + s(r-\bar{r}) \cdot F[K(t), L(t)] = f_1$

(5) $L(t+1) = L(t) + n\left(\frac{F(t)}{L(t)}\right) \cdot L(t) = f_2.$

We assume that

(6. 1) $\frac{\partial K(t+1)}{\partial K(t)} = \frac{\partial f_1}{\partial K} = 1 + s' \frac{\partial r}{\partial K} \cdot F + sF_K > 0$

(6. 2) $\frac{\partial K(t+1)}{\partial L(t)} = \frac{\partial f_1}{\partial L} = s' \frac{\partial r}{\partial L} \cdot F + sF_L > 0$

(6. 3) $\frac{\partial L(t+1)}{\partial K(t)} = \frac{\partial f_2}{\partial K} = n' \frac{\partial (F/L)}{\partial K} \cdot L > 0$

(6. 4) $\frac{\partial L(t+1)}{\partial L(t)} = \frac{\partial f_2}{\partial L} = 1 + n' \frac{\partial (F/L)}{\partial L} \cdot L + n > 0.$

The difference equations of the above system satisfy the conditions of nonnegative partial derivatives and homogeneity of the first degree, and hence the economic system represented by these difference equations will generate a balanced growth path, provided that (6. 1) and (6. 4) hold true with strict inequalities. Thus, on the balanced growth path, capital, labor and output will grow at the same constant growth rate and the path is globally stable. There is no force that will change the values of the output-capital and output-labor ratios.

Define the Jacobian at the point of the balanced path as:

(7) $$J^* = \begin{bmatrix} \left(\frac{\partial f_1}{\partial K}\right)^* & \left(\frac{\partial f_1}{\partial L}\right)^* \\ \left(\frac{\partial f_2}{\partial K}\right)^* & \left(\frac{\partial f_2}{\partial L}\right)^* \end{bmatrix}$$

where $\left(\frac{\partial f_1}{\partial K}\right)^*$ and etc. are the equilibrium (or balanced) path values of (6. 1) – (6. 4). The balanced growth rate of this economy is equal to the largest (positive) characteristic root $\lambda(J^*)$ ——the Frobenius root of

(8) $|\lambda I - J^*| = 0.$

Since J^* is an indecomposable and stable matrix, the Frobenius root $\lambda(J^*)$ is greater than the other root in its absolute value (Morishima [5]) and, hence, the balanced path is relatively stable.

Consider next the case of an economy in which aggregate production of income depends on n different capital goods and m different qualities of labor inputs, subject to a linear and homogeneous production function (see Sato and Koizumi [7]);

(9) $Y = F[a_1K_1, a_2K_2, \cdots, a_nK_n, b_1L_1, b_2L_2, \cdots, b_mL_m]$

where K_i = capital good of age i, $i = 1, \cdots, n$,

L_j = labor input of the j^{th} quality $j = 1, \cdots, m$,

a_i = fraction of K_i used for the production of Y, $i = 1, \cdots, n$, $1 \geq a_i \geq 0$,

b_j = fraction of L_i used for the production of Y, $j = 1, \cdots, m$, $1 \geq b_j \geq 0$.

The production function satisfies the usual conditions of positive marginal productivities with respect to all factor inputs and the convexity of isoquants, *i. e.,*

$$(10.1) \quad \frac{\partial F}{\partial K_i} > 0 \quad i = 1, \cdots, n, \quad \frac{\partial F}{\partial L_j} > 0 \quad j = 1, \cdots, m$$

(10. 2) B is a negative definite matrix,

where B is equal to

$$B = \begin{bmatrix} 0 & \frac{\partial F}{\partial K_i} & \frac{\partial F}{\partial L_j} \\ \frac{\partial F}{\partial K_i} & F_{kr} & \\ \frac{\partial F}{\partial L_j} & & \end{bmatrix}$$

And F_{kr} are cross partial derivatives of F with respect to K_i and L_j, $k=K_1, \cdots, K_n, L_1, \cdots, L_m$ and $r=K_1, \cdots, K_n, L_1, \cdots, L_m$.

It should be noted that the capital good of age zero K_0 and the labor input of quality zero L_0 are not included in the production function. ΔK_0 is a net addition of investment out of current income Y and K_0 has zero marginal productivity in the production of the current year's output. In the same way L_0, which is merely the total population does not enter in the production processes until L_0 is transformed to a skilled labor L_1, L_2, etc.

The essential idea behind the above assumption is that *capital has a gestation period before it is available for production.* Without loss of generality, we can assume that, with a proper choice of time unit, the gestation period is equal to one. That is to say, it takes at least one unit of time interval, before a capital good becomes "productive." In the same way it takes at least one period before the crude labor forces become trained or skilled so that they may be productive.

A constant fraction[1)] of current income is saved and invested, i. e.,

$$(11) \quad K_0(t+1) = K_0(t) + sF[K_1, \cdots, K_n L_1, \cdots, L_m] \quad 0 < s < 1,$$

where $K_0(t)$ is the current amount of the 0^{th} age capital good. An alternative interpretation of $K_0(t)$ is that we can look at $K_0(t)$ as the total amount of intermediate goods and it does not enter into the production of current output directly. In this model we do not simply assume that $K_0(t)$ automatically becomes $K_1(t)$ as time goes on at the end of the current year. Instead, $K_1(t)$ is generated by a combination of $K_0, \cdots, K_n$ as well as $L_1, \cdots, L_m$. For the sake of simplicity we assume linear technology, i. e.,

$$(12) \quad K_1(t+1) = k_{10}K_0(t) + k_{11}K_1(t) + \cdots + k_{1n}K_n(t)$$
$$+ l_{11}L_1(t) + l_{12}L_2(t) + \cdots + l_{1m}L_m(t)$$
$$k_{1r} > 0, \qquad r = 0, 1, \cdots, n$$

1) Theoretically a constant fraction of income saved assumption can be easily generalized to a variable saving ratio assumption.

$$l_{1j} > 0, \quad j = 1, \cdots, m.$$

The coefficinets k_{1r} and l_{1j} represent the various productivity coefficients of K_{1r} and L_{1j} and also the proportion of these factors used in the production of the $K_1(t+1)$ factor input. Once a "productive" capital K_1 is produced, its productivity will be further increased as time goes on until it reaches a maximum (or optimal) time period τ.

Thus:

$$(13)\quad \begin{aligned} K_2(t+1) &= k_{21}\cdot K_1(t), & k_{21} &\geq 1 \\ K_3(t+1) &= k_{32}\cdot K_2(t), & k_{32} &\geq 1 \\ &\vdots & &\vdots \\ K_\tau(t+1) &= k_{\tau\tau-1}K_{\tau-1}(t), & k_{\tau\tau-1} &\geq 1. \end{aligned}$$

However, as we extend the period of production beyond τ, the depreciation and obsolescence effects set in, which make the net productivity of $K_{\tau+1}$ smaller than that of K_τ. And, finally at the age of K_{n+1} the marginal productivity drops to zero and its employment ceases. Hence we have,

$$(14)\quad \begin{aligned} K_{\tau+1}(t+1) &= \delta_1 K_\tau(t) \\ K_{\tau+2}(t+1) &= \delta_2 K_{\tau+1}(t) \\ &\vdots \qquad\qquad 0 < \delta_i \leq 1 \quad (i = 1, \cdots n-\tau) \\ K_n(t+1) &= \delta_{n-\tau} K_{n-1}(t). \end{aligned}$$

By the assumption K_{n+1} (and $K_{n+2}\cdots$) has zero marginal productivity, either because it physically ceases to be capital or it cannot be economically used as a capital good, i. e., $\delta_{n-\tau+j}=0$, $j=1, \cdots$. Figure 1 illustrates the marginal productivity conditions of various age distributions of capital.

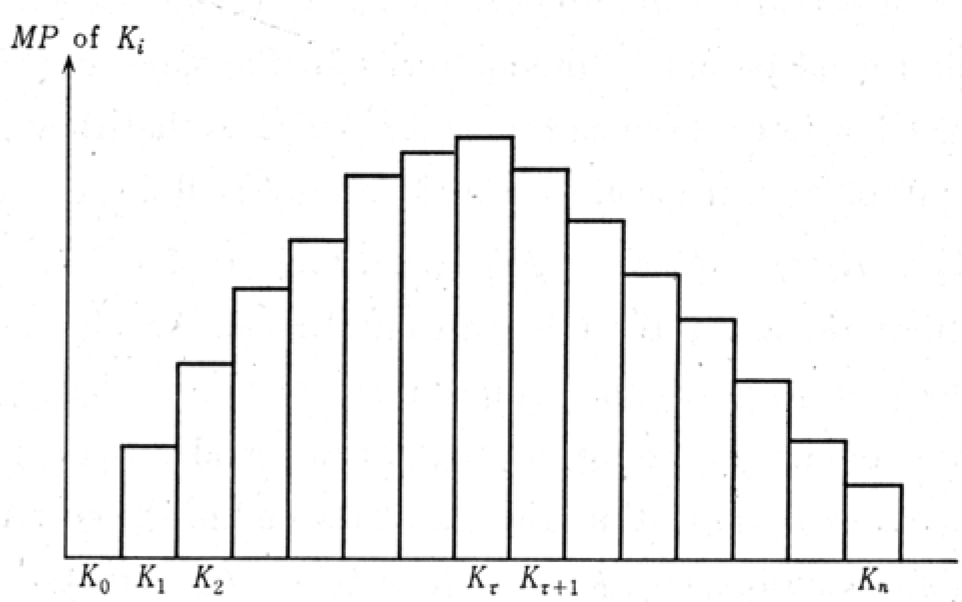

Figure 1

The labor force in this model is generated by a Malthusian-like population growth function

$$(15)\quad L_0(t+1) = (1+n[(1-s)\,Y/L_0])\,L_0(t).$$

The total population at time $t+1$, $L_0(t+1)$ depends on per capita consumption[2]. In the present

2) In endogenous growth models if population depends on per capita consumption rather than per capita income, then the golden rule of accumulation is different (see Davis [3]).

model the total population does not, as it is, participate in the production of income. Instead, only skilled (in some sense) labor after a certain period of training and education, participates in the production process. Again, without loss of generality we can choose the time unit such that L_1 is the minimum quality of the labor force that has a positive marginal productivity in the production of aggregate income Y. Once the labor force achieves this minimum level, then the learning process begins in such a way that $L_2, \cdots, L_\theta$ have higher productivities than L_1. (L_θ producing the highest.) Then gradually the $L_{\theta+1}, \cdots, L_m$ levels of the labor force become less productive due to old age, sickness, etc. (A man may be most productive at the age of say, 45 and then gradually becomes less productive.) As the labor force reaches at L_{m+1}, then it no longer serves as a factor input and retires from productive processes. These assumptions concerning the effects of training and education are summarized in the following equations:

$$
\begin{aligned}
L_1(t+1) &= \kappa_{10}K_0(t)+\kappa_{11}K_1(t)+\cdots+\kappa_{1n}K_n(t) \\
&\quad +\lambda_{10}L_0(t)+\lambda_{11}L_1(t)+\cdots+\lambda_{1m}L_m(t) \\
&\qquad \kappa_{1r}>0 \quad r=0,1,\cdots,n, \qquad \lambda_{1p}>0 \quad p=0,\cdots,m. \\
L_2(t+1) &= \lambda_{21}L_1(t), \qquad \lambda_{21}\geq 1 \\
L_3(t+1) &= \lambda_{32}L_2(t), \qquad \lambda_{32}\geq 1 \\
&\vdots \\
L_\theta(t+1) &= \lambda_{\theta\theta-1}L_{\theta-1}(t), \qquad \lambda_{\theta\theta-1}\geq 1 \\
L_{\theta+1}(t+1) &= \rho_1 L_\theta(t), \qquad 0<\rho_1\leq 1 \\
L_{\theta+2}(t+1) &= \rho_2 L_{\theta+1}(t), \qquad 0<\rho_2\leq 1 \\
&\vdots \\
L_m(t+1) &= \rho_{m-\theta}L_{m-1}(t), \qquad 0<\rho_{m-\theta}\leq 1 \\
L_{m+1}(t+1) &= \rho_{m-\theta+1}L_m(t), \qquad \rho_{m-\theta+1}=0.
\end{aligned}
\tag{16}
$$

The marginal productivities of L_j ($j=1, \cdots, m$) will have the same shape as those of capital goods (the same as Figure 1).

Note that this model is sufficiently general in that even if we allow for the possibility of nonlinear technology for aging and training, we have substantially the same results. Also the possibility of changing one's occupation which requires training and education, can be easily brought in the model by simply adjusting the assumption that $\kappa_{1r}, \lambda_{1p}, \cdots, \rho_1, \cdots, \rho_{m-\theta}$ are no longer constant, but they are homogeneous functions of zero degree with respect to $K_0, \cdots, K_n$, $L_0, \cdots, L_m$. In this way the model can handle a wide variety of different cases.

II. *The Stability Properties of the Model*

We first summarize the model:

$$
\begin{aligned}
Y(t) &= F[a_1K_1(t), a_2K_2(t), \cdots, a_nK_n(t), b_1L_1(t), b_2L_2(t), \cdots, b_mL_m(t)] \\
K_0(t+1) &= K_0(t)+sF[K_1(t), K_2(t), \cdots, K_n(t), L_1(t), L_2(t), \cdots, L_m(t)] = H^0 \\
K_1(t+1) &= k_{10}K_0(t)+k_{11}K_1(t)+\cdots+k_{1n}K_n(t) \\
&\quad +l_{11}L_1(t)+l_{12}L_2(t)+\cdots+l_{1m}L_m(t) = H^1
\end{aligned}
$$

$$
\begin{aligned}
K_2(t+1) &= k_{21}K_1(t) && = H^2 \\
&\vdots && \vdots \\
K_\tau(t+1) &= k_{\tau\tau-1}K_{\tau-1}(t) && = H^\tau \\
K_{\tau+1}(t+1) &= \delta_1 K_\tau(t) && = H^{\tau+1} \\
K_{\tau+2}(t+1) &= \delta_2 K_{\tau+1}(t) && = H^{\tau+2} \\
&\vdots && \vdots \\
K_n(t+1) &= \delta_{n-\tau}K_{n-1}(t) && = H^n \\
L_0(t+1) &= (1+n[(1-s)Y(t)/L_0(t)])L_0(t) && = H^{n+1} \\
L_1(t+1) &= \kappa_{10}K_0(t)+\kappa_{11}K_1(t)+\cdots+\kappa_{1n}K_n(t) && \\
&\quad +\lambda_{10}L_0(t)+\lambda_{11}L_1(t)+\cdots+\lambda_{1m}L_m(t) && = H^{n+2} \\
L_2(t+1) &= \lambda_{21}L_1(t) && = H^{n+3} \\
L_3(t+1) &= \lambda_{32}L_2(t) && = H^{n+4} \\
&\vdots && \vdots \\
L_\theta(t+1) &= \lambda_{\theta\theta-1}L_{\theta-1}(t) && = H^{n+\theta+1} \\
L_{\theta+1}(t+1) &= \rho_1 L_\theta(t) && = H^{n+\theta+2} \\
L_{\theta+2}(t+1) &= \rho_2 L_{\theta+1}(t) && = H^{n+\theta+3} \\
&\vdots && \vdots \\
L_m(t+1) &= \rho_{m-\theta}L_{m-1}(t). && = H^{n+m+1}.
\end{aligned}
$$

Balanced growth is defined to mean

$$
\begin{aligned}
& Y(t+1) = gY(t), \\
(17)\quad & K_i(t+1) = gK_i(t), \qquad i = 0, 1, \cdots, n \\
& L_j(t+1) = gL_j(t), \qquad j = 0, 1, \cdots, m \\
& \qquad g = 1+n^* > 0
\end{aligned}
$$

where n^* is the balanced growth rate of population. The above condition in turn implies that

$$
\begin{aligned}
& Y(t) = \alpha \overline{Y} g^t, \\
(18)\quad & K_i(t) = \alpha V_i g^t, \qquad i = 0, 1, \cdots, n \\
& L_j(t) = \alpha V_{n+1+j} g^t, \qquad j = 0, 1, \cdots, m.
\end{aligned}
$$

Since the $Y(t)$, $K_i(t)$, $L_j(t)$ are essentially positive, we must be able to choose values of α, $\overline{Y}$, V_i, V_{n+1+j} which are all positives. Inserting $K_i(t)=\alpha V_i g^t$ and $L_j(t)=\alpha V_{n+1+j} g^t$ in the H^k functions and using the homogeneity of these functions gives

$$(19)\quad gV_k = H^k(V_0, \cdots, V_{n+m+1}), \qquad (k = 0, \cdots, n+m+1)$$

The constant α can be adjusted to insure that $\sum_{k=0}^{n+m+1} V_k=1$, and we suppose this done.

The existence of a solution to the above, and thus the possibility of balanced growth in (1), is not immediately apparent, but can be proved as follows. The vector $V=(V_0, \cdots, V_{n+m+1})$ with $V_k \geq 0$, $\sum V_k=1$, defines a closed simplex in Euclidian $(n+m+1)$ space. Consider the points $y=(y_0, \cdots, y_{n+m+1})$ determined by

$$(20) \quad y_s = \frac{H^s(V_0, \cdots, V_{n+m+1})}{\sum_{k=0}^{n+m+1} H^k(V_0, \cdots, V_{n+m+1})} \qquad s=0, 1, \cdots, n+m+1$$

Clearly $y_s \geq 0$, $\sum y_k = 1$, so that we have here a continuous mapping of the closed simplex into itself. According to the fixed-point theorem of Brouwer, such a mapping has a fixed-point. That is, there is at least one vector V^* carried into itself by this transformation of coordinates. For this vector V^* we can write

$$(21) \quad V_s^* \sum H^k(V_0^*, \cdots, V_{n+m+1}^*) = H^s(V_0^*, \cdots, V_{n+m+1}^*) \qquad s = 0, 1, \cdots, n+m+1.$$

Thus V^* is a solution to our problem, with

$$g = \sum H^k(V_0^*, \cdots, V_{n+m+1}^*).$$

Thus:

proposition 1: For proper initial conditions, a steady geometric growth is always possible for the model defined above. Should K_i and L_j (also Y) once find itself in the proportions

$V_0^* : V_1^* : \cdots : V_{n+m+1}^*$,

it will remain in these proportions and be multiplied by the factor g each new unit of time. The economy grows at the constant rate of $n^* = g - 1$.

We can prove the uniqueness of the solutions V_k^* directly in the same way as Solow and Samuelson [8] and Fisher [4], but we leave the task to the reader. Instead, we shall investigate indirectly the uniqueness property and stability property by studying the Jacobian of the model in the equilibrium point. The Jacobian of the system at the balanced growth path is shown in the following page.

proposition 2: The Jacobian J^* has the maximum positive characteristic root $\lambda(J^*)$ and greater than the absolute values of any other characteristic roots. The maximum root $\lambda(J^*)$ is unique and equal to the expansion factor of the balanced growth path, $g = \lambda(J^*) = 1 + n$.

Proof: J^* is an indecomposable nonnegative matrix and in view of the Perron-Frobenius theorem there exists a maximum positive characteristic root $\lambda(J^*)$ greater than or equal to the absolute values of any other characteristic roots. In addition, J^* is a stable matrix in the sense that $\lim_{t \to \infty} \frac{J^{*t}}{\lambda^t(J^*)} y$ exists with $y \geq 0$, since J^* is indecomposable and has at least one row with all positive elements. Hence $\lambda(J^*)$ must be greater than (not equal to) the absolute values of any other characteristic roots.

Proposition 3: The balanced growth path is relatively stable with any positive initial conditions.

Proof: This proposition is a direct consequence of the last proposition. Since there exist no other characteristic roots greater or equal to $\lambda(J^*)$, which is the largest, the system must grow at the expansion rate $\lambda(J^*)$ and must be relatively stable. The economy starting from any positive quantities must eventually approach the balanced growth path given sufficient time.

$$
\begin{bmatrix}
1 & sF^*_{K_1} & sF^*_{K_2} & \cdots & sF^*_{K_n} & 0 & sF^*_{L_1} & sF^*_{L_2} & \cdots & sF^*_{L_m} \\
k_{10} & k_{11} & k_{12} & \cdots & k_{1n} & 0 & l_{11} & l_{12} & \cdots & l_{1m} \\
0 & k_{21} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & k_{32} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots k_{\tau\tau-1} \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots 0 \quad \delta_1 \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots 0 \quad 0 \;\, \delta_2 \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots \quad \delta_{n-\tau} & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & n'(1-s)F^*_{K_1} & n'(1-s)F^*_{K_2} & \cdots & n'(1-s)F^*_{K_n} & \begin{matrix}(1+n)\\ -n'(1-s)\dfrac{Y}{L_0}\end{matrix} & n'(1-s)F^*_{L_1} & & \cdots & n'(1-s)F^*_{L_m} \\
\varkappa_{10} & \varkappa_{11} & \varkappa_{12} & \cdots & \varkappa_{1n} & \lambda_{10} & \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\
0 & 0 & 0 & \cdots & 0 & 0 & \lambda_{21} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \lambda_{32} & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \lambda_{\theta\theta-1} \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \quad \rho_1 \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \quad 0 \cdots \rho_{m-\theta} & 0
\end{bmatrix}
$$

One important special case occurs when population growth is exogenous (or the case of the limitational factor input). That is to say,

$$
(22) \quad L_0(t+1) = (1+n)L_0(t), \qquad n = \text{const.}
$$
$$
n > 0.
$$

This is the standard assumption in the Solow-Swan neoclassical growth model of homogeneous capital and labor inputs. And this assumption may be more relevant to advanced countries such as U. S. and Western European countries. Under certain assumptions we can still derive conditions (sufficient conditions) for balanced growth, i. e., $1+n=g$.

Under this assumption (equation (22)), the Jacobian J^* must be changed to

$$
\begin{array}{cc}
\quad\quad L_0(t)^{th} \text{ Column} & \\
J^* = \begin{bmatrix} & \Delta_1 & \\ 0 \;\; 0 \;\; \cdots & (1+n) & 0 \;\; \cdots \;\; 0 \\ & \Delta_2 & \end{bmatrix} & L_0(t+1)^{th} \text{ row}
\end{array}
$$

where Δ_1 and Δ_2 are exactly the same as before. Since the values of the characteristic roots are unchanged by the permutation of a column and the corresponding row, we can write J^* as $\bar{J}^*$

$$\bar{J}^* = \begin{bmatrix} & & 0 \\ & & \vdots \\ & & 0 \\ & A^* & \lambda_{10} \\ & & \vdots \\ & & 0 \\ 0\ 0 & \cdots\ 0\ 0 & (1+n) \end{bmatrix}$$

where A^* stands for the Jacobian of $(n-1)\times(n-1)^{th}$ order other than the $L_0(t)^{th}$ column and $L_0(t+1)^{th}$ row. (The $L_0(t)^{th}$ column and the $L_0(t+1)^{th}$ row are deleted from the original Jacobian.)

Proposition 4: *If the expansion factor is greater than or equal to the maximum row sum of A^*, then the economy has a unique balanced growth path in which the growth rate of the entire economy is equal to the exogenous growth rate of population. That is to say, if*

$$(23) \quad (1+n) \geq \operatorname*{Max}_i \Big(\sum_{j=1}^{n-1} a_{ij}\Big)$$

then $1+n=g$, where a_{ij} are the elements of A^*.

Proof: A^* is an indecomposable and stable nonnegative matrix and, hence, there exists the greatest positive root $\lambda(A^*)$. In view of the relationship between the maximum row sum and this root, we have

$$\operatorname*{Max}_i \Big(\sum_{j=1}^{n} a_{ij}\Big) > \lambda(A^*) > \operatorname*{Min}_i \Big(\sum_{j=1}^{n-1} a_{ij}\Big).$$

On the other hand, $(1+n)$ is one positive characteristic root of $\bar{J}^*$ or J^*. In order that $(1+n)$ may be the largest root we must have $(1+n) \geqq \operatorname*{Max}_i \big(\sum_{j=1}^{n-1} a_{ij}\big) > \lambda(A^*)$. This proves the proposition.

The economic interpretation of this proposition is that if the expansion factor (Von Neumann factor) is greater than or equal to the maximum value of the sum of productivity factors, then the economy achieves balanced growth. In a special case of homogeneous capital and labor inputs (the Solow-Swan case), (23) reduces to

$$(1+n) \geqq 1 + \frac{\partial Y}{\partial K}.$$

That is to say, *the growth rate of the economy must be greater than or equal to the rate of interest,* a well-known proposition.

We can further sharpen Proposition 4, by utilizing a theorem by Ostrowski and Schneider [6].

Proposition 5: *If the following condition is met, then the economy with exogenous labor input will achieve balanced growth:*

$$(24) \quad (1+n) \geqq U$$

where $U < \max_i \left(\sum_{j=1}^{n-1} a_{ij}\right)$ and

$$U = R - \varepsilon(R-\rho); \qquad R = \max_i \left(\sum_{j=1}^{n-1} a_{ij}\right)$$

$$\varepsilon = \left(\frac{\kappa}{R-\lambda}\right)^{n-2} \qquad \rho = \frac{1}{n-1}\sum r_i$$

$$r_i = \min_i \left(\sum_{j=1}^{n-1} a_{ij}\right)$$

$$\lambda = \min_i a_{ii}$$

$$\kappa = \min_{i \neq j} a_{ij} \qquad (a_{ij} > 0).$$

Proof: (24) satisfies the Ostrowski-Schneider theorem [6]. The important thing about this proposition is that *the expansion factor* $(1+n)$ *can be less than the maximum value of the row sum. That is to say, the expansion factor can be less than the maximum value of the sum of the productivity factors.*

III. *Concluding Remarks*

In the foregoing analysis we have shown that the economic system consisting of heterogeneous factor inputs, whose supply conditions are endogenously determined, can generate a stable growth path like the standard neoclassical two-factor homogeneous input case. The important conclusions emerging from the propositions of the previous sections are that the "changes" in the marginal productivities of factor inputs will change the "growth rate" of the economy, and also that, since the supply conditions are determined by the interactions of various sectors, the balanced growth rate is proportionately related to the *number* of factor inputs, and thus to the *number* of sectors in the economy. In other words, if the economy discovers ways to either slow down the obsolescence effects or to increase the productivities of factor inputs, thereby increasing the number of production-participating sectors, then the balanced growth rate of the overall economy can be increased. This conclusion is especially pertinent to a developing country in which the intersectoral linkage effects (round-about methods of production) will determine the growth performance of the economy.

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(Brown University, U. S. A.)

投稿規程

本誌は，1962年7月発行の第13巻3号で紙面の一部を研究者の自発的な投稿制による原稿のために割くことを公表いたしましたが，それ以来，かなりの数の研究者の投稿を経て今日にいたりました。ここに改めて本誌が投稿制を併用していることを明らかにし，投稿希望を募ります。投稿規定は次のとおりです。

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