UNCERTAINTY AND THE LIFE CYCLE THEORY OF SAVING

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Analyzing the lifetime allocation process of a consumer to whom a life insurance-annuity scheme is available, Yaari [3 p. 147] had set forth a proposition that "the introduction of insurance is equivalent to the removal of uncertainty from the allocation problem" (not in terms of the level, but in terms of the basic time shape, of consupmtion).

The mechanics is straightforward. Consider an individual who starts his life at time O with zero initial assets and who possesses a utility function which is time-invariant and concave everywhere. Let T be the time of death, which is a stochastic variable defined on $[O, \bar{T}]$, where $\bar{T}$ is a non-stochastic number representing a theoretical maximum age he can ever attain. Also let $w(t)$ be his human income; $c(t)$ be his consumption; $r(t)$ be the interest rate; $\delta(t)$ be the subjective discount rate ($r(t)$ and $\delta(t)$ measured net of the biological uncertainty); and $s(t)$ be his expected survival rate such that $s(O)=1$, $s(\bar{T})=0$ and $s'(t)<0$ for all $t \varepsilon [O, \bar{T}]$. Taking the simpler case of no bequest motive, the consumer maximizes his *expected* lifetime utility sum subject to the constraint that the *expected* lifetime earnings be equal to his expected lifetime consumption, i. e.,

$$(1) \quad \underset{\{c_t\}}{\text{Maximize}} \int_O^{\bar{T}} u(c_t) \exp\left\{-\int_O^t \delta(x)\,dx\right\} s(t)\,dt$$

subject to

$$(2) \quad \int_O^{\bar{T}} [w(t)-c(t)] \exp\left\{-\int_O^t r(x)\,dx\right\} s(t)\,dt=0,\ c(t) \geqq 0$$

If there is no biological uncertainty (as well as economic uncertainty which Yaari ignores altogether), $s(t) \equiv 1$ and this problem yields a well known solution that whenever $c(t)$ is positive,

$$(3) \quad d \ln c_t/dt=[1/\varepsilon(t)][r(t)-\delta(t)] : \varepsilon(t)=d \ln u'(c_t)/d \ln c$$

In the presence of biological uncertainty, write

$$s(t)=\exp\left\{\int_O^t s'(x)/s(x)\,dx\right\}$$

Using this, (1) and (2) can be rewritten as

$$(1)' \quad \underset{\{c_t\}}{\text{Maximize}} \int_O^{\bar{T}} u(c_t) \exp\left\{-\int_O^t \bar{\delta}(x)\,dx\right\}$$

subject to

$$(2)' \quad \int_O^{\bar{T}} [w(t)-c(t)] \exp\left\{-\int_O^t \bar{r}(x)\,dx\right\} dt=0,\ c(t) \geqq 0$$

where $\bar{r}(t)=r(t)-s'(t)/s(t)$ and $\bar{\delta}(t)=\delta(t)-s'(t)/s(t)$ are the "biological risk-inclusive"

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interest rate and discount rate, respectively. Since (1)′ and (2)′ are formally identical to the certainty case, and since $\bar{r}(t)-\bar{\delta}(t)=r(t)-\delta(t)$, the solution still satisfies (3). Thus the only difference between the two cases is the *level* of consumption. Setting $\varepsilon=1$ for simplicity, the level of the initial consumption for the certainty case is given by

$$(4) \quad c_o=\frac{\int_o^T w(t)\exp\left\{-\int_o^t r(x)\,dx\right\}dt}{\int_o^T \exp\left\{-\int_o^t \delta(x)\,dx\right\}dt}$$

whereas that for the uncertainty case is give by

$$(4)' \quad \bar{c}_o=\frac{\int_o^T w(t)\exp\left\{-\int_o^t \bar{r}(x)\,dx\right\}dt}{\int_o^T \exp\left\{-\int_o^t \bar{\delta}(x)\,dx\right\}dt}$$

Clearly, c_o and $\bar{c}_o$ are generally different from each other, the difference being crucially dependent on the time shape of human income $\{w(t)\}$, for given r, δ and s. So the above statement of Yaari has been confirmed.

The economics involved is equally simple. First, rewrite the global constraint (2)′ in the form of a *flow* constraint:

$$(5) \quad r(\upsilon).\,a(\upsilon)+w(\upsilon)=[r(\upsilon)-s'(\upsilon)/s(\upsilon)]a(\upsilon)+w(\upsilon)=c(\upsilon)+\dot{a}(\upsilon)$$

where $a(\upsilon)$ is the asset position of the individual at age υ, i. e.,

$$(6) \quad a(\upsilon)=\int_o^\upsilon [w(t)-c(t)]\exp\left\{\int_o^t \bar{r}(x)\,dx\right\}dt$$

Second, note that the term $-s'(\upsilon)/s(\upsilon)>0$ is the instantaneous "probability" of dying at age υ. Since it is assumed that the individual derives no utility from leaving a bequest whereas he is not permitted to die with negative assets, the insurance contract is such that the insurance company agrees to "cancel" the individual's assets upon his death. Consider an individual whose asset position is negative at age υ. To bear the risk of having to bring his asset position to zero upon his death, the insurance company will charge a competitive premium equal to the (negative) asset value times the probability of death. The quantity $-a(\upsilon).\,s'(\upsilon)/s(\upsilon)<0$ will thus measure the subtraction from the individual's income which he can dispose for consumption and saving. This is precisely what equation (5) describes. If the asset position is positive, on the other hand, the quantity $-a(\upsilon).\,s'(\upsilon)/s(\upsilon)>0$ will be the bonus paid to the individual by the insurance company in return for his agreement to give up his assets upon death. Since the individual has no bequest motive, he will never hold "regular notes" which yeild only $r(t)$. On the other hand, if the asset position is negative, he is certainly willing to pay the premium because of the assumed prohibitively high penalty on dying with negative assets. The constraint (2)′ says that if "acturial notes" are thus the only type of assets the individual holds, then he will behave, in the presence of such an insurance-annuity scheme, as if there were no uncertainty except that the interest and discount rates are now inclusive of biological uncertainty. Since the $s(t)$ function is free from stochasticity, so is the maximization problem.

A minor complication arises when the individual has a bequest motive. In this case, he will not be content with complete cancellation of assets upon his death, but will wish to leave some assets to his family. If one sticks to the complete cancellation rule, the only form of leaving bequests is to hold regular notes. This implies holding of a mixed portfolio consisting of two types of assets yielding different rates of return. Yaari showed, however, that in this case, too, the basic time shape of consumption will be the one described in (3). Even if one assumed a more realistic type of insurance contracts in which the insurance company agrees to pay a specific positive amount of money $b(t)$ upon death, the same result would obtain, provided that the $b(t)$ function is independent of the asset position $a(t)$[1].

II

We have just seen that the introduction of insurance in fact removes uncertainty from the allocation process. With insurance, the problem is reduced to a deterministic decision problem in spite of the fact that the individual will never know when he will die. The important implication of this deterministic property is that the problem need be solved once at the start of life and that no intermediate revisions are necessary.

However, it is the basic fact of life that we are exposed to a variety of uncertainty and risk, some of which are essentially uninsurable. The important category of uncertainty in the present context is the *economic* uncertainty, i. e., uncertainty about future human incomes and future interest rates[2]. Take, for exmple, uncertainty about future human incomes. The first decision problem an individual faces is that of choosing an occupation. If he is rational, he will choose that occupation which gives him the highest expected present value of lifetime earnings among all the alternatives open to him. In this calculation, he would first take an observed representative income stream for each occupation and then apply certain discounts to allow for the uncertainty elements due to competition, future economic changes, etc. Such discounting due to uncertainty is not only theoretically justifiable but is in agreement with empirical evidence[3]. The point is that the observed figures which abstract from uncertainty elements are *not* relevant in the calculation of an individual who is launching into the world. What are relevant are the figures net of uncertainty elements. Moreover, such uncertainty elements will continue to exist even after he has chosen an occupation, though there are reasons to believe that they decline as he gains experience.

Let us suppose that the individual faces, in addition to biological uncertainty, uncertainty

1) In such a case, the flow constraint would be given by $\bar{r}(t)a(t)+w(t)=c(t)+\dot{a}(t)-b(t)s'(t)/s(t)$. So long as $b(t)$ is not a function of $a(t)$, the existence of the last term on the right-hand side would not affect the basic result.

2) Strotz [1] has dealt with the problem in which the subjective discount rate is variable over lifetime but its lifetime profile is not known in advance. We shall, however, continue to assume that such subjective changes are fully anticipated at the start of life. Nonetheless, our result will be formally quite similar to his.

3) It seems to provide an explanation, for example, to observed differentials in earnings among different occupations and among different regions or sectors in an economy.

about his future human incomes (while still abstracting from uncertainty about future interest rates)[4]. The existence of such uncertainty will immediately be reflected in his calculation of the present values of future human income. Let us denote the discount rate due to this economic uncertainty by $\sigma(t)$. Now the individual discounts his future income $w(t)$ not at the rate $\bar{r}(t)$, but at the rate $\bar{\bar{r}}(t)=\bar{r}(t)+\sigma(t)$[5]. Consider an individual living at age υ. The present value of his lifetime earnings which he can spend for consumption in the remainder of his life is

$$(7) \quad a(\upsilon)+\int_0^T w(t)\exp\left\{-\int^t \bar{\bar{r}}(x)\,dx\right\}dt,$$

where $a(\upsilon)$ is the current asset position as resulting from his past activity, and given by

$$(8) \quad a(\upsilon)=\int_0^{\upsilon}[w(t)-c(t)]\exp\left\{\int_t^{\upsilon}\bar{r}(x)\,dx\right\}dt$$

If the individual has no bequest motive, the expression in (7) must be equated to the present value of future consumption

$$(9) \quad \int_{\upsilon}^T c(t)\exp\left\{-\int_{\upsilon}^t \bar{r}(x)\,dx\right\}dt$$

Suppose we perform the maximization calculation of the following problem

$$(10) \quad \underset{\{c_t\}}{\text{Maximize}} \int_{\upsilon}^T u(c_t)\exp\left\{-\int_{\upsilon}^t \bar{\delta}(x)\,dx\right\}dt$$

subject to

$$(11) \quad a(\upsilon)+\int_{\upsilon}^T w(t)\exp\left\{-\int_{\upsilon}^t \bar{\bar{r}}(x)\,dx\right\}dt=\int_{\upsilon}^T c(t)\exp\left\{-\int_{\upsilon}^t \bar{r}(x)\,dx\right\}dt \text{ and } c(t)\geqq 0.$$

Assuming a simple logarithmic utility function $u=\ln c$ we get:

$$(12) \quad c_{\upsilon}(t)=\lambda\exp\left\{\int_{\upsilon}^t[\bar{r}(x)-\bar{\delta}(x)]\,dx\right\};\ (\bar{T}\geqq t\geqq \upsilon\geqq 0)$$

where

$$(13) \quad \lambda=c_{\upsilon}(\upsilon)=\frac{a(\upsilon)+\int_{\upsilon}^T w(t)\exp\left\{-\int_{\upsilon}^t \bar{\bar{r}}(x)\,dx\right\}dt}{\int_{\upsilon}^T \exp\left\{-\int_{\upsilon}^t \bar{\delta}(x)\,dx\right\}dt}$$

Equation (12) says that the consumption plan laid down at an arbitrary intermediate time υ still satisfies equation (3). However, the important difference is this. With $\sigma(t)>0$, or $\bar{\bar{r}}(t)>\bar{r}(t)$,

4) This is not to deny the importance of uncertainty about future rates of return. But for the present illustrative purpose, this source of uncertainty is not essential.

5) The notion of $\sigma(t)$ needs some defense. The second term of (7) can be written as

$$\int_{\upsilon}^T w(t)\exp\left\{-\int_{\upsilon}^t \bar{r}(x)\,dx\right\}\frac{R(t)}{R(\upsilon)}dt$$

where $R(t)$, $t\varepsilon[O, T]$, is the prior subjective prospect of realizing $w(t)$ at time t and related to $\sigma(t)$ by $\sigma(t)=-R'(t)/R(t)$. $R(t)/R(\upsilon)$ $(t>\upsilon)$ measures, therefore a similar prospect but conditional upon the fact that he has realized $w(\upsilon)$. As υ increases, i. e. as the individual gains experience, one can expect that $R(t)/R(\upsilon)$ increases (toward unity) for any prescribed future date t. In terms of a more general stochastic model, our $w(t)$ represents an expected value taken over the conditional probability density function $f(w_t/\text{all past values of } w)$, and $\sigma(t)$ represents a measure of variance also taken over the same conditional probability density function. But in this paper we avoided explicit treatment of stochastic elements by assuming (1) that expected $w(t)$ is all realized, and (2) that $R(t)$ is a prescribed decreasing function of time.

$c_v(v)$ is *not* equal to $c_0(O)$. $\exp\left\{\int_0^v [\bar{r}(x)-\bar{\delta}(x)]dx\right\}$, as one can easily see. This means that it pays to revise the consumption plan at intermediate points, rather than sticking to the initially set profile all the way through. One way of looking at the issue is to examine the flow constraint implied by equation (11). Differentiating (11) with respect to time yields the following flow constraint:

$$(11)' \quad w(v)+\bar{r}(v).\, a(v)=c(v)+\dot{a}(v)+\sigma(v).\, \exp\left\{-\int_v^t \bar{\bar{r}}(x)dx\right\}dt$$

The left-hand side is the total earnings at time v, whereas the right-hand side shows the disposition of the total earnings. The first two terms are obvious. It is the last term that is new. This term is clearly non-negative, and represents the individual's subjective reservation due to uncertainty about future earnings. Although he is aware that the *ex post* constraint will be free from this term, in his *ex ante* calculation, he behaves as if his total earnings at time v were $w(v)+r(v).\, a(v)-\sigma(v).\int_v^T w(t)\exp\left\{-\int_v^t \bar{\bar{r}}(x)dx\right\}dt$ rather than $w(v)+\bar{r}(v).\, a(v)$. That is, to the extent that the present value of lifetime earnings depends on uncertain future incomes, the individual consumes less, for given $a(v)$, than he would in the absence of such uncertainty.

Granted that the consumption plan laid down at the start of life is not to be followed throughout lifetime, and hence that revisions become necessary, we must make an assumption about the frequency with which such revisions are made. We assume in what follows that such revisions take place continually. This means that the consumption plan effective at time v is the plan laid down at the same time point v, i. e., $c_v(v)$, in our previous notation. We are interested in knowing what sort of lifetime consumption profile is implied in this continually revised plan.

To see this, differentiate $c_v(v)$ in (13) logarithmically with respect to time, and we obtain

$$(14) \quad \frac{d \ln c_v(v)}{dv}=\bar{\bar{r}}(v)-\bar{\delta}(v)-\sigma(v).\, \alpha(v)=\boxed{\bar{\bar{r}}(v)-\bar{\delta}(v)}+\sigma(v).\,[1-\alpha(v)]$$

where $\alpha(v)$ is the asset-lifetime earnings ratio given by

$$(15) \quad \alpha(v)=\frac{a(v)}{a(v)+\int_v^T w(t)\exp\left\{-\int_v^t \bar{\bar{r}}(x)dx\right\}dt}$$

In equation (14), the term within the dotted rectangle is the certainty counterpart, as we saw earlier. The term $\sigma(v)[1-\alpha(v)]$ is clearly non-negative, since $\sigma(v)>0$ and $1-\alpha(v)\geqq 0$. Thus this equation tells us that the revised consumption profile would be higher in the proportionate rate of change than the consumption profile without uncertainty. The level of consumption, on the other hand, depends crucially on the time shape of human income. We noted that the level of consumption at any time v is lower than the certainty counterpart for given $a(v)$. But since this means that assets are growing faster in the revision model than in the certainty model, and since we are assuming that the lifetime earnings are the same in retrospect for both cases, the consumption level of the present revision model will exceed that of the certainty model at least for some length of time. Clearly, the more concentrated the human income profile to early years, the larger the assets and the smaller the discount on future huamn incomes, and hence the higher the consumption level for any time point υ. The converse is also true. Hence we can draw a general conclusion that the consumption profile of the present revision model tends to resemble that of human income.

Figure 1 describes the revised consumption profile in a semilog paper. A family of dotted lines represent the slope $[\bar{r}(t)-\bar{\delta}(t)]$, and a family of semi-broken curves represent the slope $[\bar{\bar{r}}(t)-\bar{\bar{\delta}}(t)]$. As equation (12) says, the consumption profile laid down at age υ is shown by one of the dotted lines. In particular, let the line AB be the profile as planned at age 0, so that the initial consumption is chosen at A. At the next instant the plan is revised, and the profile laid down then is shown by a different dotted line, which determines the "initial" consumption at this instant, say, at C. The slope connecting A and C will be the one shown by equation (14).

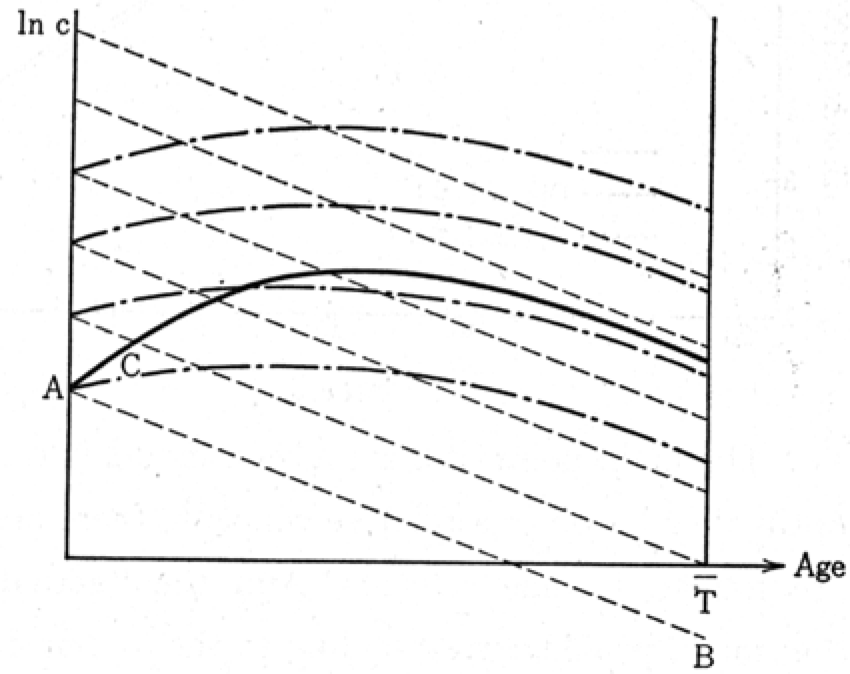

Figure 1

The solid curve starting from A describes the profile of revised consumption on the assumption that the w profile is such that the asset position is first negative and then turns positive.

III. A NOTE ON THUROW

Thurow [2] proposed an ingenious way of deriving an "optimal" consumption profile from budget data. He traced out such an optimal profile by calculating the level of income at which the average person in a particular age group neither saves nor dissaves. That is to say, his "optimal" profile corresponds to our optimal profile which would prevail if the human income stream were such that $w(\upsilon)=c(\upsilon)$ for all υ. Thurow's "optimal" profile would be described by

$$(16)\qquad \frac{d\ln c_o(\upsilon)}{d\upsilon}=\bar{\bar{r}}(\upsilon)-\bar{\bar{\delta}}(\upsilon)\ \text{ for all } \upsilon,$$

,in our notation, since $\alpha(\upsilon)=0$ at all times, if we ingore $a(O)$. A somewhat stylized picture of this optimal profile this is given by a dotted curve in Figure 2[6].

A translation of this dotted curve into a semi-log paper will give us the value $\bar{\bar{r}}(\upsilon)-\bar{\bar{\delta}}(\upsilon)$ (if ε is not unitary, $\frac{1}{\varepsilon}[\bar{\bar{r}}(\upsilon)-\bar{\bar{\delta}}(\upsilon)]$) for all υ. From Figure 2, we observe that $\bar{\bar{r}}(\upsilon)-\bar{\bar{\delta}}(\upsilon)<0$ almost everywhere with a systematic tendency to decline with age. The solid curve, on the other hand, shows the "actual" consumption profile, which corresponds to our equations (13) and (14). Actual consumption starts low reflecting a concentration of human income in distant

6) Figure 2 is based on Thurow's Figure 2 [2, p. 328]. In his Figure 2, Thurow shows two optimal profiles corresponding to different definitions of consumption. The dotted curve shown in our Figure 2 corresponds to his optimal profile #1 in which consumption is measured by expenditures for current consumption (excluding

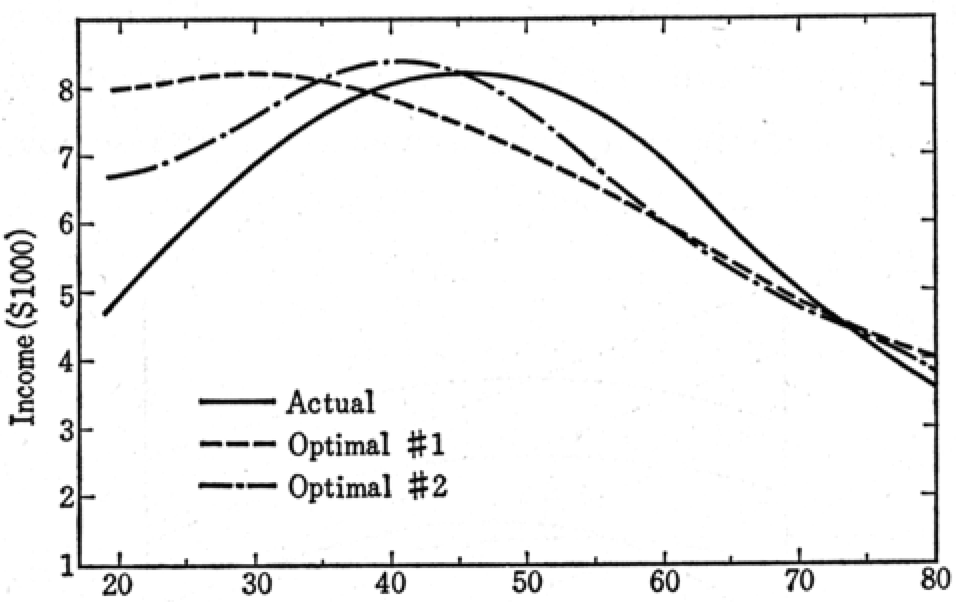

Figure 2

future, but keeps rising faster for a while due to negative asset position. In the fifties the asset position turns positive. In this later phase, actual consumption declines faster. Thurow does not report the corresponding average income stream and asset positions. But it seems fair to say that it is the human income profile that causes a similar "hump" in the actual consumption profile.

The main point of our revision model is this. In the presence of such economic uncertainty, optimality becomes a relative concept; there are as many optimal consumption profiles as there are human income streams. And the "actual" consumption profile can be explained as an "optimal" profile corresponding to the human income profile that prevails in our economy.

IV. SUMMARY

The fundamental result of the existing life cycle theory of saving is that the basic time shape of optimal consumption profile is independent of that of human income. But this leaves a systematic gap between the theory and reality, and this gap is customarily attributed to imperfections in the capital market which prevent free reallocation of lifetime earnings (see, e. g., Thurow, *ibid.,* p. 329).

The present paper is an attempt to explain, within an equally simple and stylized framework, why consumers behave as they actually do. In explaining this, an emphasis has been laid on the economic uncertainty which is inherent in individuals intertemporal decision making but which is essentially uninsurable. It has been shown (1) that in the presence of such uncertainty revisions in the consumption plan become essential; (2) that the revised consumption becomes dependent on the human income profile; (3) that the dependence of revised consumption on the human income profile is such that the consumption profile tends to resemble that of humain income; and (4) that it seems possible to interpret observed consumption profile as one such revised optimal consumption profile corresponding to the actual lifetime pattern of human income.

REFERENCES

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gifts, contributions and personal insurance payments). His optimal profile #2 corresponds to consumption gross of terms in parenthesis and shown by the semi-broken curve. For exposition, I shall use optimal profile #1.