EXISTENCE OF STATIONARY EQUILIBRIUM IN THE WALRAS-WICKSELLIAN MODEL OF PRODUCTION*

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Léon Walras' theory of capital formation [6, Section V] can be examined in several ways. One point that remains ambiguous is whether or not his theory necessarily presupposes a progressive economy where net addition to capital and net saving are taking place.[1] Walras himself said that his economy is a progressive one, but it was Wicksell who pointed out that this was wrong and Walras' theory would be more appropriately applicable to a stationary economy, thereby rather gaining "in rigor" [7, pp. 226–7]. In fact a recent article by M. Morishima [4] reveals that the economic world Walras dealt with may be not only a progressive economy, but also a stationary or even a retrogressive one, because in his system there is no definite relationship between gross investment and depreciation of capital. The aim of this paper is to recast the Walrasian general equilibrium system of capital formation into a model of a strictly *stationary* economy, and to prove, with respect to this model, the existence of equilibrium. In building our model we take account of some aspects of Wicksell's version of a stationary state that has been reformulated by R. Solow [5] and simplified a bit by R. Dorfman [2]. So ours might be named the Walras-Wicksellian model of production or (gross) capital formation.

* This is a slightly revised version of my paper which was written under the stimulus of Morishima's significant article [4] during my stay at Stanford University in 1960–61. I am indebted to Professors Kenneth J. Arrow, Michio Morishima, Hirofumi Uzawa and especially to Ken-ichi Inada for very valuable comments and suggestions. Any error which may remain is of course the author's whole responsibility.

1) As for other ambiguities, see [3].

1. We begin with some preliminary remarks which are necessary to the construction of our model.

Suppose there are n kinds of commodities in an economy, the first group of which consists of m kinds of consumers' goods labeled $1, \cdots, m$, and the second group $n-m$ kinds of capital goods labeled $m+1, \cdots, n$. Outputs of the first group of commodities are taken for pure final goods in the sense that they are only for use of households, never serving as inputs in production. Outputs of the second group of commodities are taken for pure intermediate products in the sense that they are entirely used as means of production, but can never serve as final consumables. These intermediate products or capital goods, however, are regarded here as durable-use goods whose span of life is finite. Thus any i-th capital good lasts N_i years,[2] whereupon it disintegrates. For brevity we shall call consumers' and capital goods as *foods* and *machines* respectively.

Since we are assuming stationary conditions, the stock of each machine, its annual rate of depreciation and its annual rate of output must all be constant. This implies that the rate at which each machine wears out and has to be replaced is constant and equal to its annual rate of output. It is evident that, in order to ensure this condition, the stock of each type of machine must be constituted in a specific manner. Thus if we denote the total stock of any i-th machine in a given year by k_i, k_i has to be evenly distributed as to age from age 1 to age N_i, and its annual output, denoted by y_i, must equal k_i/N_i.

Next we turn to the structure of production. Be-

2) 'year' is taken as a unit of time period for convenience.

sides n kinds of commodities let there be s kinds of primary factors, labeled 1, ⋯, s. We shall call them s kinds of *labor*. We represent by

$a_{ij}(i=m+1, \cdots, n;\ j=1, \cdots, n)$ the amount of the i-th machine service needed to produce one unit of the j-th commodity which may be either a food or a machine. Similarly, $b_{ij}(i=1, \cdots, s;\ j=1, \cdots, n)$ represents the amount of the i-th labor per unit of the j-th commodity. The production coefficients a_{ij}'s and b_{ij}'s are, as usual, assumed to be non-negative constants.

In stationary equilibrium the total stock of every machine must be fully utilized. Hence we must have, for any i-th machine,

$$\sum_{j=1}^{m} a_{ij}x_j + \sum_{j=m+1}^{n} a_{ij}y_j = k_i, \quad i=m+1, \cdots, n,$$

where x_j's stand for the respective annual output of each kind of food. It is assumed here that one unit of machines of each type offers one unit of its service per year, so that the stock of each type of machine and the amount of its service per year can be expressed by the same number.

On the other hand it has been already observed that the relation $y_i = k_i/N_i$ holds for any type of machine in the stationary conditions. Hence it follows that

$$\text{(i)} \qquad \sum_{j=1}^{m} a_{ij}x_j + \sum_{j=m+1}^{n} a_{ij}(k_j/N_j) = k_i, \ i=m+1, \cdots, n.$$

This is the form that is needed hereafter.

Let P_j be the price of a new machine of the j-th type, r_j be its annual rental (*i. e.*, the price of its service). Then we have the following well-known relationship between P_j and r_j,

$$P_j = r_j \frac{1-(1+i)^{-N_j}}{i}, \quad j=m+1, \cdots, n,$$

where i is the annual rate of interest and r_j is assumed to be independent of time. Barring the case where the rate of interest is zero, the above equation could be rewritten as

$$\text{(ii)} \qquad P_j = qr_j\left[1-\left(\frac{q}{1+q}\right)^{N_j}\right], \quad j=m+1, \cdots, n,$$

where q is the reciprocal of the rate of interest, i. e., $q \equiv 1/i$. Since y_i or k_i/N_i is the annual output of the i-th machine, the total gross investment in a given year is $\sum_{i=m+1}^{n} P_i y_i$ or $\sum_{i=m+1}^{n} qr_i\left[1-\left(\frac{q}{1+q}\right)^{N_i}\right]\frac{ki}{N_i}$. This gross investment must be matched by gross saving. Following Walras, we define gross saving, i. e., the excess of gross income over consumption, as qz, the product of q and z, where z is the demand for interest which would be earned from gross saving, and is assumed to be a well-defined function of the variables specified later. It will be easily seen that our demand for interest is analogous to the demand for "marchandise idéale" (E) which Walras mentioned in his theory of capital formation and credit.[1)]

2. We are now in a position to write down our model of stationary equilibrium in matrix form. At the outset the notation is defined in what follows:

$x=(x_1, \cdots, x_m)$ is a column vector whose i-th component represents the annual output of the i-th food. Similarly

$k=(k_{m+1}, \cdots, k_n)$ is a column vector for the stocks of the machines.

$L=(L_1, \cdots, L_s)$ is a column vector for the total supply of each kind of labor.

$p=(p_1, \cdots, p_m)$ is a row vector for the prices of the foods. Similarly

$P=(P_{m+1}, \cdots, P_n)$ is a row vector for the prices of the machines.

$r=(r_{m+1}, \cdots, r_n)$ is a row vector for the annual rentals of the services of machine.

$w=(w_1, \cdots, w_s)$ is a row vector for the annual wages of each kind of labor.

N_i, a scalar, is the length of life of the i-th machine, and assumed to be a given positive integer.

As explained above, q is the reciprocal of the rate

1) See [6], esp. 250.

of interest, and z is the demand for interest.

Finally we need the following six matrices:

$$A_1=\begin{bmatrix} a_{m+1,1}\cdots\cdots a_{m+1,m} \\ \cdots\cdots\cdots\cdots \\ a_{n1}\cdots\cdots a_{nm} \end{bmatrix},\quad A_2=\begin{bmatrix} a_{m+1,m+1}\cdots\cdots a_{m+1,n} \\ \cdots\cdots\cdots\cdots \\ a_{n,m+1}\cdots\cdots a_{nn} \end{bmatrix},$$

$$B_1=\begin{bmatrix} b_{11}\cdots\cdots b_{1m} \\ \cdots\cdots\cdots\cdots \\ b_{s1}\cdots\cdots b_{sm} \end{bmatrix},\quad B_2=\begin{bmatrix} b_{1,m+1}\cdots\cdots b_{1n} \\ \cdots\cdots\cdots\cdots \\ b_{s,m+1}\cdots\cdots b_{sn} \end{bmatrix},$$

$$N=\begin{bmatrix} N_{m+1} & & \\ & \ddots & \\ & & N_n \end{bmatrix},\quad V=\begin{bmatrix} 1-\left(\frac{q}{1+q}\right)^{N_{m+1}} & & \\ & \ddots & \\ & & 1-\left(\frac{q}{1+q}\right)^{N_n} \end{bmatrix}.$$

Note that A_2 is a square matrix while N and V are diagonal ones.

Our system of equations can then be formulated as follows:

(1) $$A_1x+A_2N^{-1}k=k,$$
(2) $$B_1x+B_2N^{-1}k=L,$$
(3) $$rA_1+wB_1=p,$$
(4a) $$rA_2+wB_2=P,$$
(4b) $$P=qrV.$$

(1) and (2) state the equality between the supply and demand of the machine-service and labor respectively. (3) and (4a) state respectively that the price of each food or machine must equal its unit costs. (4b) is simply a generalization of (ii) above, which expresses the equality between the price of a machine of each type and the discounted present value of its rentals over the whole length of its life. It is more convenient to put together (4a) and (4b) and obtain a single equation

(4) $$rA_2+wB_2=qrV.$$

Furthermore there are the demand and supply functions, relating x, L and z to p, w, r, q and k respectively, i. e.,

(5) $$x=x(p, w, r, q, k),$$
(6) $$L=L(p, w, r, q, k),$$
(7) $$z=z(p, w, r, q, k).$$

Finally we have an equation which expresses the equality between gross saving and gross investment

(8) $$qz=qrVN^{-1}k.$$

All that is necessary to complete our system is some consideration of Walras' law. In our system this law is given by

$$px+qz\equiv rk+wL$$

which is an identity in any non-negative p, w, r, q and k such that $(p, q)\neq 0$ and $(w, r)\neq 0$.

3. Our problem is now to prove the

Theorem: *Except an economically trivial case in which nothing is produced of all kinds of the foods, the system (1)-(8) possesses a non-negative solution.*

To show this, the following assumptions are used:

(a) The demand and supply functions x, L and z are respectively single-valued and continuous functions of $p\geqq 0$, $w\geqq 0$, $r\geqq 0$, $q\geqq 0$ and $k\geqq 0$ such that $(p, q)\neq 0$ and $(w, r)\neq 0$. x and L are always non-negative for these values of the variables, and are positively homogeneous of zero degree with respect to p, w and r. z is positively homogeneous of first degree with respect to the same variables.

(b) The supply functions L are bounded so that to each component of L there corresponds a least upper bound respectively.

(c) The supply functions L are such that L_i (= the i-th component of L) >0 implies $w_i>0$. This is equivalent to saying that $w_i=0$ implies $L_i=0$.

(d) The demand function z is such that $z\leqq 0$ for $p\neq 0$, $q\geqq \bar{q}$, where $\bar{q}$ is a sufficiently large positive number.

(e) (i) Every row and (ii) every column of A_1 has at least one positive element.

(f) A_2 is indecomposable.

(g) (i) Every row and (ii) every column of B_1 and B_2 has at least one positive element.[1)]

Proof of the theorem: p can be always chosen so

1) Mathematically there is no need of the assumption that every column of B_1 has at least one positive element. We preserve, however, this assumption for economic reasons.

as to make the following equation hold:

$$rA_1+wB_1=p,$$

hence p_i's are not independent variables and could be eliminated. Note that p, determined by this equation, is always semi-positive ($p\geq 0$) by virtue of $r\geqq 0$, $w\geqq 0$ and $(w, r)\neq 0$, and of assumptions (ei) and (gi).

Construct the following expressions:

$$(9)\qquad E(w, r, q, k)\equiv A_1x+A_2N^{-1}k-k$$

$$(10)\qquad F(w, r, q, k)\equiv B_1x+B_2N^{-1}k-L$$

$$(11)\qquad G(w, r, q)\equiv qrVN^{-1}-rA_2N^{-1}-wB_2N^{-1}$$

$$(12)\qquad H(w, r, q, k)\equiv z-rVN^{-1}k.$$

Then we have

$$(13)\qquad rE+wF+Gk+qH=px+qz-rk-wL\equiv 0.$$

Let the domains of k and q be $0\leqq k_i\leqq \bar{k}_i$, $(i=m+1, \cdots, n)$, and $0\leqq q\leqq \bar{q}$ respectively, where $\bar{k}_i$'s are chosen as arbitrary positive numbers so large that, for each $(\bar{k}_i)$, $i=m+1,\cdots,n$, at least one component of $B_2N^{-1}(\bar{k}_i)$ is greater than the least upper bound of the corresponding component of L.[1] Here $(\bar{k}_i)$ is defined as a column vector $(0, \cdots, \bar{k}_i, \cdots, 0)$, i. e., a column vector whose i-th component is $\bar{k}_i$, while all of the other components are zero. Next we normalize w and r so as to make their component sum equal one, i. e., $r_i\geqq 0$, $w_j\geqq 0$, $\sum r_i+\sum w_j=1$.

We define

$$E^*=\max(0, E),\qquad F^*=\max(0, F),$$
$$G^*=\max(0, G),\qquad H^*=\max(0, H),$$

and consider the following continuous mapping:[2]

$$(14a)\qquad R_i=\frac{r_i+E_i^*}{1+S^*},\qquad i=m+1, \cdots, n,$$

$$(14b)\qquad W_j=\frac{w_j+F_j^*}{1+S^*},\qquad j=1, \cdots, s,$$

$$(14c)\qquad K_i=\frac{k_i+\bar{k}_iG_i^*}{1+S^*+G_i^*},\qquad i=m+1, \cdots, n,$$

$$(14d)\qquad Q=\frac{q+\bar{q}H^*}{1+S^*+H^*},$$

1) Note that $B_2N^{-1}(\bar{k}_i)\geq 0$ owing to assumption (gii).

2) Our mapping technique and the ensuing discussions are a slight modification of Morishima's. See [4].

where $S^*\equiv\sum E_i^*+\sum F_j^*$.

We see that $R_i\geqq 0$, $W_j\geqq 0$, $\sum R_i+\sum W_j=1$, $0\leqq K_i\leqq \bar{k}_i$ and $0\leqq Q\leqq \bar{q}$. Therefore, owing to Brouwer's theorem, there exists a fixed point $(\hat{r}, \hat{w}, \hat{k}, \hat{q})$ such that

$$(15a)\qquad \hat{r}_i=\frac{\hat{r}_i+\hat{E}_i^*}{1+\hat{S}^*},$$

$$(15b)\qquad \hat{w}_j=\frac{\hat{w}_j+\hat{F}_j^*}{1+\hat{S}^*},$$

$$(15c)\qquad \hat{k}_i=\frac{\hat{k}_i+\bar{k}_i\hat{G}_i^*}{1+\hat{S}^*+\hat{G}_i^*},$$

$$(15d)\qquad \hat{q}=\frac{\hat{q}+\bar{q}\hat{H}^*}{1+\hat{S}^*+\hat{H}^*}.$$

First we prove that $\hat{E}^*$, $\hat{F}^*$, $\hat{G}^*$ and $\hat{H}^*$ are all zero. Suppose at least one component of $\hat{E}^*$ or $\hat{F}^*$ to be positive so that $\hat{S}^*>0$. Then it can be shown that

$\hat{r}_i>0$ *if and only if* $\hat{E}_i^*>0$,
$\hat{w}_j>0$ *if and only if* $\hat{F}_j^*>0$,
$\hat{k}_i>0$ *if and only if* $\hat{G}_i^*>0$,
$\hat{q}>0$ *if and only if* $\hat{H}^*>0$,

whence it follows $\hat{r}\hat{E}+\hat{w}\hat{F}+\hat{G}\hat{k}+\hat{q}\hat{H}>0$, a contradiction to (13). Therefore we must have $\hat{S}^*=0$, that is to say, $\hat{E}_i^*=0$ and $\hat{F}_j^*=0$, and of course $\hat{E}^*=0$ and $\hat{F}^*=0$.

Suppose next at least one component of $\hat{G}^*$, say $\hat{G}_i^*$, to be positive. Since $\hat{S}^*=0$, we obtain from (15c) $\hat{k}_i\hat{G}_i^*=\bar{k}_i\hat{G}_i^*$. Thus $\hat{k}_i=\bar{k}_i$. $\bar{k}_i$ is, however, chosen so large as to make at least one component of $B_2N^{-1}(\bar{k}_i)$ greater than the least upper bound of the corresponding component of L. Hence at least one component, say the s-th, of $B_1\hat{x}+B_2N^{-1}\hat{k}$ is greater than $\hat{L}_s$, the s-th component of $\hat{L}$, so that $\hat{F}_s^*>0$ follows. This contradicts $\hat{F}^*=0$. Therefore $\hat{G}_i^*=0$, and of course $\hat{G}^*=0$.

Finally, suppose $\hat{H}^*>0$. Since $\hat{S}^*=0$, we get from (15d) $\hat{q}\hat{H}^*=\bar{q}\hat{H}^*$ Thus $\hat{q}=\bar{q}$. Then the assumption (d) shows that $\hat{z}\leqq 0$ (note that $\hat{r}A_1+\hat{w}B_1=\hat{p}\geq 0$), and so $\hat{H}=\hat{z}-\hat{r}\hat{V}N^{-1}\hat{k}\leqq 0$, a contradiction. Therefore $\hat{H}^*=0$.

We have now proved that there exists a fixed point such that $\hat{E}\leqq 0$, $\hat{F}\leqq 0$, $\hat{G}\leqq 0$ and $\hat{H}\leqq 0$.

Our remaining problem is to eliminate the inequality signs from these expressions. To do this, let us assume that $\hat{x}$ is not zero, i. e., semi-positive. (If $\hat{x}$ happens to be zero and there is no other solution, our story ends here and we cannot go on any further. But this is surely an economically uninteresting as well as trivial case.)

At first it may be noticed that the above inequalities, together with $\hat{r}\hat{E}+\hat{w}F+\hat{G}\hat{k}+\hat{q}\hat{H}=0$, enable us to deduce that

(16a) *if* $\hat{r}_i>0$, *then* $\hat{E}_i=0$,

(16b) *if* $\hat{w}_j>0$, *then* $\hat{F}_j=0$,

(16c) *if* $\hat{k}_i>0$, *then* $\hat{G}_i=0$,

(16d) *if* $\hat{q}>0$, *then* $\hat{H}=0$.

Now the expression $E\leqq 0$ shows that $A_1\hat{x}\leqq(I-A_2N^{-1})\hat{k}$. But $A_1\hat{x}\geq 0$ because of our assumption $\hat{x}\geq 0$ and of assumption (eii). Thus $A_2N^{-1}\hat{k}\leq\hat{k}$ holds for a $\hat{k}\geqq 0$. On the other hand, A_2 is indecomposable owing to assumption (f), and so is A_2N^{-1}. Hence, by a theorem of Debreu-Herstein's [1, p. 602], the maximal characteristic root of A_2N^{-1} is less then one, so that $(I-A_2N^{-1})^{-1}>0$. Then we have $(I-A_2N^{-1})^{-1}A_1\hat{x}\leqq\hat{k}$, of which the left-hand side is clearly positive. Thus $\hat{k}>0$. By (16c) this implies $\hat{G}=0$.

Next, since $\hat{F}\leqq 0$ we have $B_1\hat{x}+B_2N^{-1}\hat{k}\leqq\hat{L}$. But the condition $\hat{k}>0$ together with assumption (gi) indicates that $B_2N^{-1}\hat{k}>0$. Hence $\hat{L}>0$, and through assumption (c) we can conclude that $\hat{w}>0$. By (16b) this implies $\hat{F}=0$.

We have shown above that $\hat{G}$ is zero. This gives $\hat{q}\hat{r}\hat{V}=\hat{r}A_2+\hat{w}B_2$ or $\hat{r}\hat{V}(\hat{q}I-\hat{V}^{-1}A_2)=\hat{w}B_2$. $\hat{w}>0$ is already known, and so $\hat{w}B_2>0$ by virtue of assumption (gii). We see at once that $\hat{q}$ is positive. (Otherwise we have a contradiction.) By (16d) this implies $\hat{H}=0$.

Furthermore $\hat{r}\hat{V}(\hat{q}I-\hat{V}^{-1}A_2)>0$ holds for a $\hat{r}\hat{V}\geqq 0$, and $\hat{V}^{-1}A_2$ is indecomposable. Hence the maximal characteristic root of $\hat{V}^{-1}A_2$ is less than $\hat{q}$ and the inverse of $(\hat{q}I-\hat{V}^{-1}A_2)$ is positive. Therefore $\hat{r}\hat{V}=\hat{w}B_2(\hat{q}I-\hat{V}^{-1}A_2)^{-1}>0$, which enables us to conclude that $\hat{r}>0$. By (16a) this implies $E=0$.

What remains is trivial. Since $\hat{r}$ is positive and assumption (eii) holds (or since $\hat{w}$ is positive and assumption (gii) holds), $\hat{r}A_1+\hat{w}B_1$ is clearly positive so that $\hat{p}>0$. Moreover it can be easily shown that $\hat{z}$ is also positive.

Summing up, we have been able to prove the above-mentioned theorem which could be now sharpened as follows:

Provided that $\hat{x}$ is not zero, the equations (1)-(8) are satisfied for $\hat{L}>0$, $\hat{z}>0$, $\hat{p}>0$, $\hat{w}>0$, $\hat{r}>0$, $\hat{q}>0$ and $\hat{k}>0$.

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